Poincaré and the Theory of Automorphic Functions

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Introduction

Poincaré's work in the theory of automorphic functions is a beautiful example of how one simple idea could unite and clarify results in different areas of mathematics; and he has left a dramatic account of the circumstances under which he conceived of the principal ideas which underlie this theory (see his essay, Mathematical Creation, reproduced elsewhere in this issue). One of the major accomplishments of the theory of automorphic functions is, of course, that it eventually led to the proof of '*Fermat's Last Theorem*' by Andrew Wiles (see *Resonance*, Vol. 1, No. 1, pp. 71-79). At the time young Poincaré embarked on his brilliant career in mathematics the notion of automorphic functions was 'in the air' in many different areas of mathematics. His first papers on '*Fuchsian functions*' (Poincaré's term for automorphic functions; see *Box* 1) set off the process of crystallisation of the theory of automorphic functions. The purpose of this article is to give a brief account of Poincaré's ideas which he alludes to in his essay.

The theory of automorphic functions in which Poincaré did pioneering work led eventually to the proof of 'Fermat's last theorem'.

Motivation

The problem proposed by the Academy of Sciences of Paris in 1880 for the prize in mathematical sciences was '*to perfect, in some important point, the theory of linear differential equations of a single variable*'. Poincaré entered his memoir which consisted of two parts, the second of which was on *Fuchsian functions*; this contains reflections inspired by his reading of a paper of L Fuchs. Poincaré had received this paper at the beginning of May 1880 and he submitted his own paper at the Academy on 28 May, 1880, barely a couple of weeks later! (The prize ultimately went to Halphen; Poincaré's paper received an honourable mention.)

To put things in perspective let us start with the problem of finding the length of an arc of a circle. This leads us to computing an integral of the form

$$f(a) = \int_0^a \frac{dy}{\sqrt{1-y^2}}.$$

As for the function $x = f(a)$, we know that its inverse function, which is nothing but $\sin x$, has nice properties like periodicity, $\sin(x + 2n\pi) = \sin x$ for any integer n, and the law of addition, $\sin(x_1 + x_2) = \sin x_1 \cos x_2 + \sin x_2 \cos x_1$.

Similarly, when one tries to determine the length of an arc of an ellipse one is led to considering integrals of the form

$$g(x) = \int_0^x \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}}.$$

This integral cannot be expressed in terms of the elementary functions known till then like sin, cos, log, ., and many mathematicians worked on them without making much headway (like for instance, Legendre) before Abel and Jacobi realised in the 1820's that it may be better to study the inverse of the function, $g^{-1}(z) = x$, than the function, $g(x) = z$, itself. It turns out that the inverses of functions defined by such integrals are *doubly periodic*, i.e., there are complex numbers ω_1, ω_2 with ω_1/ω_2 being a non-real complex number such that

$$g^{-1}(z + m\omega_1 + n\omega_2) = g^{-1}(z)$$

for integers m, n. They also satisfy an addition formula. Such functions are called *elliptic functions* just as the usual trigonometric functions are called *circular functions*.

Soon after, this method of studying inverses of functions occurring as solutions of differential equations/integral equations became popular and many new examples, *automorphic functions*, were discovered. Automorphic functions are functions with an extraordinary amount of symmetry; the elliptic functions are among simpler examples.

For instance, H A Schwarz discovered the *triangle functions* while studying the solutions of the *hypergeometric* differential equations. These were functions defined, to begin with,

Box 1.

Poincaré was motivated by the paper of Fuchs and so in his paper he (somewhat hastily, perhaps?) named the groups and functions he considered after Fuchs; he later identified a class of groups as *Kleinian*. Klein and Poincaré had spirited exchanges regarding these names but neither could convince the other and in the end Poincaré and the French school stuck to using these personal names while the followers of Klein used descriptive titles like *Hauptkreissgruppe* (*principal-circle-group*), etc. The name *automorphic function* seems to have been used for the first time by Klein in his paper in 1890; this name has gained wide currency and is used most widely today.

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on a triangle inside the unit circle with lines or arcs of circles orthogonal to the unit circle as sides; they were then continued as *analytic functions* to the whole of the unit disk by reflecting the triangle along the sides till they fill up the unit disk (i.e., the triangle and its various reflections form a *tessellation* of the disk) and using the *Riemann–Schwarz reflection principle.*

Fuchs had similarly considered linear second order differential equations and obtained results similar to those of Schwarz. On reading this paper of Fuchs, Poincaré considered the problem of finding figures bounded by lines or arcs of circles inside the unit disk whose reflections would form a tessellation, of the unit disk. Though it had been realised that the groups of *fractional linear transformations* or *Möbius transformations* were what were involved in the reflection process giving rise to the tessellation, what was lacking was a *geometric* understanding of the situation. It was Poincaré's inspiration to realise that the *non-Euclidean* geometry discovered by Bolyai and Lobachevsky, also known as *hyperbolic* geometry, held the key.

Poincaré's Model of the Hyperbolic Plane

Consider the *upper half-plane* $\mathbf{H} = \{x + iy \in \mathbf{C} \ : \ y > 0\}$. This is conformally equivalent to the interior of the unit disk $\mathbf{D}^1 = \{z \in \mathbf{C} \ : \ |z| < 1\}$ by the map $z \mapsto (z - i)/(z + i)$ which takes i to the origin and so the triangle functions, tessellation, etc., can be considered in the upper half-plane instead of in the unit disk.

The Möbius transformations on $\mathbf{H}$ are nothing but the action of the group $SL(2, \mathbf{R})$: if σ is a matrix with real entries and determinant 1 and $z \in \mathbf{H}$ then σ acts on $\mathbf{H}$ by $\sigma(z) = (az + b)/(cz + d)$ where

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(Exercise: Check that $(az + b)/(cz + d) \in \mathbf{H}$.) Since

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad -\sigma = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$$

represent the same Möbius transformation $z \mapsto (az+b)/(cz+d)$ it is natural to identify σ and $-\sigma$ and consider

$$PSL(2,\mathbf{R}) = SL(2,\mathbf{R})/\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

as acting on $\mathbf{H}$.

If we declare the straight lines perpendicular to the x-axis and arcs of circles with centres on the x-axis as *straight lines* or *geodesics* in $\mathbf{H}$ we get Poincaré's model of the hyperbolic plane. The usual *hyperbolic metric*, $ds = \sqrt{dx^2 + dy^2}/y$, and the *volume element*, $dxdy/y^2$, are invariant under the action of $SL(2,\mathbf{R})$ and Poincaré observed that $PSL(2,\mathbf{R})$ becomes the group of isometries of $\mathbf{H}$.

To get tessellations of $\mathbf{H}$ we need to consider *discontinuous subgroups* of $SL(2,\mathbf{R})$, i.e., roughly speaking, subgroups Γ of $SL(2,\mathbf{R})$ which have no limit points (in Γ); the subgroup $SL(2,\mathbf{Z})$ of matrices with integer coefficients and determinant 1 is an example of a discontinuous subgroup. Such groups are examples of *Fuchsian groups.* In a series of papers which followed rapidly Poincaré showed how to construct a *fundamental domain* for a Fuchsian group; a fundamental domain with its images under the action of the Fuchsian group will give a tessellation of $\mathbf{H}$. For the group $SL(2,\mathbf{Z})$ a fundamental domain is

$$\mathcal{F} = \{z \in \mathbf{H} : |\mathbf{Re}(z)| \leq \frac{1}{2}, |z| \geq 1\}.$$

(Exercise: Draw a picture of $\mathcal{F}$ and its image under the transformation $z \mapsto -1/z$.)

Suppose Γ is a Fuchsian group. A meromorphic function $f : \mathbf{H} \to \mathbf{C}$ is called a *Fuchsian function* for Γ if

$$f((az+b)/(cz+d)) = f(z)$$

for all

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \quad z \in \mathbf{H}.$$

The problem which Poincaré encountered immediately was whether for a given Fuchsian group Fuchsian functions exist. His proof of existence of Fuchsian functions for a given group was inspired by Jacobi's expression of elliptic functions as quotients of theta functions. He constructed what are now known as *Poincaré series* for any given Fuchsian group and thus solved the problem of existence of Fuchsian functions.

Poincaré's researches in this subject led him also to a fundamental theorem known as the uniformisation theorem (see *Box* 2).

Suggestions for Further Study

Our account of automorphic functions has been very sketchy with only the historical aspects in mind. Another shortcoming of this account has been the lack of figures; fundamental domains of Fuchsian groups and the associated tessellations make very beautiful pictures. The reader is urged to look into the books of Ford and Lehner. One needs a good background in complex analysis to study automorphic functions. We would highly recommend the following classic:

L V Ahlfors, *Complex Analysis*, McGraw-Hill, 1979.

For a nice treatment of the theory of automorphic functions

Box 2. Uniformisation.

On the circle $x^2 + y^2 = 1$, the variable y is a two-valued function $\sqrt{1 - x^2}$ of x. One 'uniformises' the curve by expressing x and y as single-valued functions of a third variable t; e.g. $x = \frac{2t}{1+t^2}$, $y = \frac{1-t^2}{1+t^2}$. Uniformisation of curves makes more sense when one allows complex values of the variables. Any curve then becomes a Riemann surface – a space on which one can do complex analysis of one complex variable. Riemann surfaces of genus 0 can be uniformised by rational functions as witnessed, for example, above. It was also known before Poincaré's time that Riemann surfaces of genus 1 could be uniformised by elliptic functions. Poincaré as well as Klein conjectured that Riemann surfaces of higher genus could be uniformised by automorphic functions. The uniformisation conjecture was eventually proved both by Poincaré and Koebe in 1907.

L R Ford, *Automorphic Functions*, McGraw-Hill, New York, 1929. (This contains the classical theory.)

J Lehner, *Discontinuous groups and automorphic functions*, American Mathematical Society, 1964. (This provides a more modern account of the subject.)

The following book is the best source for the arithmetic aspects of automorphic functions:

G Shimura, *An introduction to the Arithmetic Theory of Automorphic Functions*, Princeton, 1994

For an introductory treatment with proofs and examples the best source is perhaps

J-P Serre, *A course in Arithmetic*, Narosa Publishers, New Delhi, 1979.

As a source book for the original papers of Poincaré, in English translation with an introduction, see H Poincaré, *Papers on Fuchsian Functions* (Translated by J Stillwell), Springer-Verlag, 1985.

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It's again going to be a mess when we approach 31 December 9999!