Continued Fractions for *e*

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Introduction

As Paul Erdős might have said, 'even babies' know that a finite simple continued fraction (SCF for short) always yields a rational number, while an infinite periodic SCF always yields a quadratic irrational. Many results of great beauty are known about CFs. Consider for instance the following two results quoted by Ramanujan to Hardy in his very first letter (the general pattern in each case is clear):

$$\int_0^a e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi} - \cfrac{e^{-a^2}}{2a + \cfrac{1}{a + \cfrac{2}{2a + \cfrac{3}{a + \cdots}}}},$$

and

$$\cfrac{1}{1 + \cfrac{e^{-2\pi}}{1 + \cfrac{e^{-4\pi}}{1 + \cdots}}} = \left\{ \left(\frac{5+\sqrt{5}}{2} \right)^{1/2} - \frac{\sqrt{5}+1}{2} \right\} e^{2\pi/5}$$

It is hard to resist recalling what Watson had to say [1] about such formulae:

> Such a formula gives me a thrill which is indistinguishable from the thrill which I feel when I enter the Sagriesta Nuova of Capelle Medicee and see before me the austere beauty of 'Day', 'Night', 'Evening' and 'Dawn' which Michelangelo has set over the tombs of Giuliano de' Medici and Lorenzo de' Medici.

Quoted by S Chandrasekhar in *Truth and Beauty*.

In this article, we examine some infinite CFs that converge to numbers related to e. For a quick preview, here is one such relation:

$$\frac{1}{e-1} = \cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+}}}}, \tag{1}$$

which may be written equivalently (and perhaps more elegantly) as

$$\frac{1}{e-2} = 1+\cfrac{1/2}{1+\cfrac{1/3}{1+\cfrac{1/4}{1+}}}. \tag{2}$$

This relation is easy to prove. Our main objective in this article will be to prove the following pretty but none too obvious identities:

$$\frac{e^2-1}{e^2+1} = \cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{7+}}}}, \tag{3}$$

and

$$\frac{1}{\sqrt{e}-1} = 1+\cfrac{2}{3+\cfrac{4}{5+\cfrac{6}{7+}}}, \tag{4}$$

The formal definition of a continued fraction is given in *Box* 1, and some well-known theorems concerning CFs are given in *Box* 2. We shall have occasion to use a result due to Emil Cesàro relating to limits of sequences; this is described in *Box* 3.

Box 1. Definition of Continued Fractions

Let $a_0, a_1, b_1, a_2, b_2, \ldots$ be integers. We shall use the notation $[a_0; (a_1, b_1), (a_2, b_2), (a_3, b_3), \ldots]$ to denote the number recursively defined as follows: $[a_0; (a_1, b_1)] = a_0 + a_1/b_1$, and for $n > 1$,

$$[a_0; (a_1, b_1), (a_2, b_2), \ldots, (a_{n-1}, b_{n-1}), (a_n, b_n)] =$$

$$[a_0; (a_1, b_1), (a_2, b_2), \ldots, (a_{n-1}, b_{n-1} + a_n/b_n)].$$

This will be referred to as a *general continued fraction* (GCF for short); it may also be written in the following manner:

$$a_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}}.$$

The notation $[a; b, c, d, \ldots]$, without the parentheses within the [ ], will refer to a *simple continued fraction* (SCF for short); it is defined as the GCF $[a; (1, b), (1, c), 3(1, d), \ldots]$; the numerators here are all 1. It may also be written as

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cdots}}}$$

For typographic convenience, we shall at times use the notation

$$a + \frac{1}{b+}\ \frac{1}{c+}\ \frac{1}{d+}\ \cdots$$

to denote the SCF $[a; b, c, d, \ldots]$.

Proof of Equation 1

Let the sequences $\{u_n\}$ and $\{v_n\}$ be defined as follows:

$$\left.\begin{array}{l} u_n = n!, \\ v_1 = 0, v_2 = 1, v_n = (n-1)(v_{n-1} + v_{n-2})(n \geq 2). \end{array}\right\} \quad (5)$$

Box 2. Facts About Continued Fractions

We briefly review here the main facts concerning CFs. For proofs the reader should consult the text by Hall and Knight, or the one by Hardy and Wright. ([1],[2])

1. Any rational number may be written as a finite SCF – in two possible ways. For instance, the rational number $4/13$ may be written as $[0; 3, 4]$ and also as $[0; 3, 3, 1]$.

2. By the infinite SCF $[a_0; a_1, a_2, a_3, a_4, \ldots]$ we shall mean the limit of the sequence

$$a_0, \quad [a_0; a_1], \quad [a_0; a_1, a_2], \quad [a_0; a_1, a_2, a_3], \quad \ldots,$$

provided that the limit exists; and similarly for an infinite GCF.

3. Let $a_1, a_2, a_3, \ldots$ be any infinite sequence of positive integers, and let a_0 be any integer. Then the infinite SCF $[a_0; a_1, a_2, a_3, \ldots]$ converges to a real number.

4. An infinite SCF corresponds to an irrational real number, and the representation of an irrational real number by an infinite SCF is unique.

5. A convergent infinite GCF does not necessarily converge to an irrational number. Also, uniqueness of representation does not hold.

For instance, we have the rather trivial relation

$$2 = [1; (2, 1), (2, 1), (2, 1), \ldots]$$

via the fact that 2 is one of the solutions of the equation $x = 1 + 2/x$. (Convergence is easy to show.) Likewise the equation $x = 2 + 3/x$ yields the relation $3 = [2; (3, 2), (3, 2), (3, 2), \ldots]$.

6. By simple calculation we find that for any SCF $[a_0; a_1, a_2, a_3, \ldots]$,

$$[a_0; a_1] = \frac{a_0 a_1 + 1}{a_1}, \quad [a_0; a_1, a_2] = \frac{a_0 a_1 a_2 + a_0 + a_2}{a_1 a_2 + 1}, \quad \ldots.$$

Let $[a_0; a_1, a_2, a_3, \ldots, a_n] = P_n/Q_n$ where P_n and Q_n are coprime; then P_n/Q_n is the n^{th} *convergent* to the SCF. The following very convenient formulae may be used for computing convergents:

continued ...

$$\left.\begin{array}{rcl} P_n & = & a_n P_{n-1} + P_{n-2} \\ Q_n & = & a_n Q_{n-1} + Q_{n-2} \end{array}\right\} \quad (n \geq 2).$$

More generally, the convergents P_n/Q_n to the GCF $[a_0; (a_1, b_1), (a_2, b_2), (a_3, b_3) \ldots]$ follow the recursive law displayed below.

$$\left.\begin{array}{rcl} P_n & = & b_n P_{n-1} + a_n P_{n-2} \\ Q_n & = & b_n Q_{n-1} + a_n Q_{n-2} \end{array}\right\} \quad (n \geq 2).$$

7. The following result holds for any two successive convergents of a SCF:

$$P_n Q_{n-1} - P_{n-1} Q_n = \pm 1.$$

(There is an easy proof via induction.)

8. An infinite SCF is *periodic* if the sequence $a_0, a_1, a_2; a_3, \ldots$ is ultimately periodic (that is, if $a_{n+p} = a_n$ for some $p > 0$ and for all n beyond some point). The following is known: *A periodic SCF converges to a quadratic irrational, that is, to a number of the form* $(a + \sqrt{b})/c$ *where* a, b, c *are integers* $(b, c > 0)$. For instance, the infinite SCF $[1; 2, 2, 2, \ldots]$ converges to $\sqrt{2}$, and $[1; 1, 2, 1, 2, \ldots]$ converges to $\sqrt{3}$. A particularly pretty SCF is $[1; 1, 1, 1, \ldots]$, which converges to the 'golden number' $(\sqrt{5} + 1)/2$.

   Interestingly, cubic and higher order algebraic irrationals never seem to turn up. Indeed, the SCFs for cube roots and higher order roots of rational numbers show no discernible pattern at all.

More results concerning CFs are known, and may be found in the books listed in the Suggested Reading.

We observe firstly that the two sequences follow the same recursive law, for

$$n! = (n-1)\Big[(n-1)! + (n-2)!\Big].$$

The first few values of u_n and v_n are shown in the following table:

Box 3. A Result Due to Emil Cesàro

Suppose that $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ are two infinite sequences of positive numbers such that

1 the series $A(t) = a_0 + a_1 t + a_2 t^2 + \cdots$ and $B(t) = b_0 + b_1 t + b_2 t^2 + \cdots$ have equal radii of convergence, say R, where $R > 0$; and

2 the limit of a_n/b_n as $n \to \infty$ exists, say $a_n/b_n \to s$.

If $A(t)$ and $B(t) \to \infty$ as $t \to R^-$, i.e. $t \to R$ from the left then

$$\lim_{t\to R^-} \frac{A(t)}{B(t)} = s.$$

Example. Let $a_n = n + 1$ and $b_n = 2n + 1$. The series

$$1 + 2t + 3t^2 + 4t^3 + \cdots, \qquad 1 + 3t + 5t^2 + 7t^3 + \cdots$$

have radii of convergence equal to 1, as may easily be shown. For $t = 1$ both series diverge, and for $|t| < 1$ the two series converge respectively to the quantities

$$A(t) = \frac{1}{(1-t)^2}, \qquad B(t) = \frac{1+t}{(1-t)^2}.$$

Observe that both these quantities diverge as $t \to 1$. Their ratio is equal to $1/(1+t)$, which tends to $1/2$ as $t \to 1$. Note that a_n/b_n tends to the same limit as $n \to \infty$.

n	1	2	3	4	5	6	7
u_n	1	2	6	24	120	720	5040
v_n	0	1	2	9	44	265	1854

We find that v_n/u_n is the $(n-1)^{\text{th}}$ partial sum of the series $(1/2 - 1/6 + 1/24 - \quad + (-1)^n/n! + \quad\cdot)$. The proof, which is easy and left for the reader, is via induction. It follows that $v_n/u_n \to 1/e$, and therefore that $u_n/v_n \to e$.

On the other hand, we find using the recursion relations (see *Box* 2), that u_n/v_n is the $(n-1)^{\text{th}}$ convergent of the GCF

$$[2;\ (2,2),\ (3,3),\ (4,4),\ (5,5)\ \ldots] = 2+\frac{2}{2+}\,\frac{3}{3+}\,\frac{4}{4+}\,\frac{5}{5+}\cdots$$

So the GCF equals e, and we obtain a result equivalent to (1).

The Main Result

We shall now prove that

$$\frac{e^2-1}{e^2+1} = \frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{5+}\,\frac{1}{7+}\cdots = [0;1,3,5,7,\ \ldots].$$

Let a_n/b_n denote the n^{th} convergent to the SCF $[0;1,3,5,7,\ \ldots]$. Then

$$\frac{a_1}{b_1} = \frac{0}{1},\quad \frac{a_2}{b_2} = \frac{1}{1},\quad \frac{a_3}{b_3} = \frac{3}{4},\quad \frac{a_4}{b_4} = \frac{16}{21}, \tag{6}$$

and in general, via the recursion law (item 6 in *Box* 2), for $n > 0$,

$$\left.\begin{aligned} a_{n+2} &= (2n+1)a_{n+1} + a_n \\ b_{n+2} &= (2n+1)b_{n+1} + b_n \end{aligned}\right\} \tag{7}$$

Observe that the two sequences follow the same recursion. For reasons of convenience, we define $a_0 = 1$ and $b_0 = 0$. The first few values of a_n and b_n are shown below.

n	0	1	2	3	4	5	6	7
a_n	1	0	1	3	16	115	1051	11676
b_n	0	1	1	4	21	151	1380	15331

Both sequences grow with tremendous rapidity. Thus, we have,

$$\begin{aligned} a_{20} &= 9508271497633004786551 \approx 9.51\times 10^{21}, \\ b_{20} &= 12484695980499706867555 \approx 1.25\times 10^{22} \end{aligned}$$

We now introduce the exponential generating functions

$$A(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, \qquad B(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}. \qquad (8)$$

Our first step will be to show that both series have radius of convergence equal to $\frac{1}{2}$. For this we need to show that

$$\lim_{n\to\infty} \left(\frac{a_n}{n!}\right)^{1/n} = 2, \qquad \lim_{n\to\infty} \left(\frac{b_n}{n!}\right)^{1/n} = 2.$$

Proof. By definition, we have $a_n = (2n-3)a_{n-1} + a_{n-2}$ for $n > 1$. Since $\{a_n\}$ is an increasing sequence of positive integers, it follows that a_n lies between $(2n-3)a_{n-1}$ and $(2n-2)a_{n-1}$, and therefore that

$$2n-3 < \frac{a_n}{a_{n-1}} < 2n-2 \text{ (for } n > 2\text{; equality holds when } n = 2).$$

Multiplying together the corresponding terms of $n-2$ such inequalities, we get

$$(2n-3)\cdot(2n-5)\cdots 3 < a_n < (2n-2)\cdot(2n-4)\cdots 4,$$

which yields, on division by $n!\,2^n$,

$$\frac{(2n-3)!}{2^{2n-2}(n-2)!n!} < \frac{a_n}{n!\,2^n} < \frac{1}{4n}.$$

Using Stirling's approximation for the factorial function we obtain, via 'sandwiching',

$$\lim_{n\to\infty} \left(\frac{a_n}{n!\,2^n}\right)^{1/n} = 1, \qquad (9)$$

showing that the radius of convergence of $A(x)$ is $\frac{1}{2}$. The proof for $B(x)$ is the same, and the same result is obtained. So both series have radius of convergence equal to $\frac{1}{2}$.

Exactly the same statements may be made about the derivatives $A'(x)$, $B'(x)$: both series have radius of convergence equal to $\frac{1}{2}$.

We now proceed to find a differential equation satisfied by both $A(x)$ and $B(x)$. Differentiation yields the following (in the summations shown below we have omitted the subscripts, to avoid clutter):

$$A'(x) = \sum a_{n+1} \ \frac{x^n}{n!}, \qquad A''(x) = \sum a_{n+2} \ \frac{x^n}{n!}, \quad (10)$$

with similar equations for $B'(x)$ and $B''(x)$. Now the relation $a_{n+2} = (2n + 1)a_{n+1} + a_n$ may also be written as

$$\frac{a_{n+2}}{n!} - 2 \ \frac{a_{n+1}}{(n-1)!} = \frac{a_{n+1}}{n!} + \frac{a_n}{n!}.$$

Multiplying throughout by x^n, summing over n and using (8) and (10), we find that $A(x)$ satisfies the following second order linear differential equation:

$$(1 - 2x)A''(x) = A'(x) + A(x).$$

The function $B(x)$ satisfies the same kind of equation. So both $A(x)$ and $B(x)$ are solutions of the equation

$$(1 - 2x)y'' = y' + y \quad (11)$$

in the unknown function y. Observe that (11) is a homogeneous linear equation with non-constant coefficients, and that $x = \frac{1}{2}$ is a singular point for the equation. Indeed, the solution cannot be easily carried beyond this point.

The most natural move towards solving the equation is to make the substitution $u = 1 - 2x$. We get, easily enough,

$$\frac{dy}{du} = -\frac{1}{2}y' \qquad \frac{d^2y}{du^2} = \frac{1}{4}y''$$

and the differential equation gets transformed to

$$4u\frac{d^2y}{du^2} + 2\frac{dy}{du} - y = 0. \quad (12)$$

Fortunately the general solution of this equation can be guessed. The form of the coefficient ($4u$) in the 1st term suggests that the solution will contain the term $e^{\sqrt{u}}$; for if $y = e^{\sqrt{u}}$, then

$$\frac{dy}{du} = e^{\sqrt{u}} \cdot \frac{1}{2\sqrt{u}}, \qquad \frac{d^2y}{du^2} = \frac{e^{\sqrt{u}}}{4u}\left(1 - \frac{1}{\sqrt{u}}\right) = \frac{y}{4u}\left(1 - \frac{1}{\sqrt{u}}\right)$$

It may be easily verified that $y = e^{\sqrt{u}}$ does indeed satisfy (12), and so does $y = e^{-\sqrt{u}}$. Since the equation is linear and of 2nd order, the general solution of (12) is

$$y(u) = ce^{\sqrt{u}} + de^{-\sqrt{u}}$$

where c, d are constants.

To find c and d we use the following: (a) $A(0) = 1$, $A'(0) = 0$ (the derivative being with respect to x); (b) $B(0) = 0$, $B'(0) = 1$; (c) $u = 1$ when $x = 0$. Also:

$$\frac{dA}{du} = \frac{dA}{dx}\,\frac{dx}{du} = -\frac{1}{2}\frac{dA}{dx}, \quad \text{so} \quad \left.\frac{dA}{du}\right|_{u=1} = 0,$$

and similarly,

$$\frac{dB}{du} = -\frac{1}{2}\frac{dB}{dx}, \quad \left.\frac{dB}{du}\right|_{u=1} = -\frac{1}{2}.$$

So to find A as a function of u we need to solve the equations $y(1) = 1$, $y'(1) = 0$ for c and d. We get, after some manipulation,

$$c = \frac{e^{-1}}{2}, \quad d = \frac{e}{2}, \quad \text{so } y(u) = \cosh(\sqrt{u} - 1).$$

In the case of B we need to solve the equations $y(1) = 0$, $y'(1) = -\frac{1}{2}$. We get,

$$c = -\frac{e^{-1}}{2}, \quad d = \frac{e}{2}, \quad \text{so } y(u) = \sinh(1 - \sqrt{u}).$$

Remembering that $u = 1 - 2x$ we have finally,

$$\left.\begin{aligned} A(x) &= \cosh(1 - \sqrt{1 - 2x}), \\ B(x) &= \sinh(1 - \sqrt{1 - 2x}). \end{aligned}\right\} \qquad (13)$$

We are almost at the end of our long journey. We need to find $\lim_{n\to\infty} a_n/b_n$, and we shall use Cesàro's result (see *Box* 3) with the value $R = \frac{1}{2}$. However we cannot immediately use the result, because the functions

$$\cosh(1 - \sqrt{1-2x}), \qquad \sinh(1 - \sqrt{1-2x})$$

do *not* diverge as $x \to \frac{1}{2}$ (divergence plays a crucial role in the theorem); indeed, the two functions take the values $\cosh 1$, $\sinh 1$, respectively. However all is not lost – we can work instead with the *derivatives* of these two functions, that is, with the functions

$$\frac{\sinh(1 - \sqrt{1-2x})}{\sqrt{1-2x}}, \qquad \frac{\cosh(1 - \sqrt{1-2x})}{\sqrt{1-2x}}.$$

The crucial facts are: (a) the derivatives tend to ∞ as $x \to \frac{1}{2}$; (b) the ratio of the coefficients of x^n in $A'(x)$ and $B'(x)$ is the same as the corresponding ratio for $A(x)$ and $B(x)$. Therefore the required limit is equal to the limit, as $x \to \frac{1}{2}$, of the quantity

$$\frac{\sinh(1 - \sqrt{1-2x})}{\sqrt{1-2x}} \div \frac{\cosh(1 - \sqrt{1-2x})}{\sqrt{1-2x}}.$$

This limit is easy to compute; the expression simplifies to $\tanh(1 - \sqrt{1-2x})$, which tends to $\tanh 1$ when $x \to \frac{1}{2}$. So the limit of a_n/b_n as $n \to \infty$ is $\tanh 1$; that is,

$$\frac{a_n}{b_n} \to \frac{e - e^{-1}}{e + e^{-1}} = \frac{e^2 - 1}{e^2 + 1},$$

and we have the stated result.

The result may be extended to the following: for any real number a,

$$\tanh a = \frac{a}{1+}\frac{a^2}{3+}\frac{a^2}{5+}\,\frac{a^2}{7+} \tag{14}$$

Equivalently:

$$\frac{e^{2a} - 1}{e^{2a} + 1} = \frac{a}{1+}\;\frac{a^2}{3+}\;\frac{a^2}{5+}\;\frac{a^2}{7+} \tag{15}$$

The other result (4) follows in the same way. We write a_n/b_n for the n^{th} convergent to

$$\frac{2}{3+}\frac{4}{5+}\frac{6}{7+}.$$

Then we have, for $n > 0$,

$$\left.\begin{aligned} a_{n+2} &= (2n+3)a_{n+1} + (2n+2)a_n \\ b_{n+2} &= (2n+3)b_{n+1} + (2n+2)b_n \end{aligned}\right\} \qquad (16)$$

We also have $a_0 = 1$, $a_1 = 0$, $b_1 = 0$, $b_1 = 1$ (as earlier). Going through the same motions, we find that the two functions $A(x) = \sum a_n x^n/n!$ and $B(x) = \sum b_n x^n/n!$ have radii of convergence equal to $\frac{1}{2}$, and both obey the homogeneous linear differential equation

$$(1-2x)y'' = (3+2x)y' + 2y. \qquad (17)$$

This is solved as earlier by the substitution $u = 1 - 2x$. We obtain, after some effort,

$$y = c\left(\frac{e^{-x}}{1-2x}\right) + d\left(\frac{1}{1-2x}\right)$$

for suitable constants c and d. Plugging in the boundary conditions, we find that

$$A(x) = \frac{2e^{-x}-1}{1-2x}, \qquad B(x) = \frac{1-e^{-x}}{1-2x}. \qquad (18)$$

The closed forms are valid, as earlier, only for $|x| < R$ (with $R = \frac{1}{2}$), but in this case, both expressions do diverge as $x \to \frac{1}{2}$. So we do not need to compute the derivatives; the required limit is

$$\lim_{x\to\frac{1}{2}^-} \frac{A(x)}{B(x)} = \lim_{x\to\frac{1}{2}^-} \frac{2e^{-x}-1}{1-e^{-x}} = \frac{2-\sqrt{e}}{\sqrt{e}-1}, \qquad (19)$$

by straightforward substitution. It follows that

$$1 + \frac{2}{3+}\frac{4}{5+}\frac{6}{7+} \qquad = \frac{1}{\sqrt{e}-1}.$$

The method we have used is quite general; the only qualification is that the differential equation we get should be solvable in closed from. If we had started with the CF

$$\frac{1}{1-}\frac{1}{3-}\frac{1}{5-}\frac{1}{7-} \tag{20}$$

where minus signs have replaced the plus signs in (3), then the differential equation reached is $(1-2x)y'' = y' - y$. The same substitution works ($u = 1-2x$; or, much better, $u^2 = 1-2x$). The result obtained is: $A(x) = \cos(1-\sqrt{1-2x})$ and $B(x) = \sin(1-\sqrt{1-2x})$ (with $A(x)$ and $B(x)$ defined exactly as earlier). Once again we must go to the first derivatives before applying the theorem of Cesàro, and we find finally that the CF converges to $\tan 1$. More generally we have, for any real number a,

$$\tan a = \frac{a}{1-}\;\frac{a^2}{3-}\;\frac{a^2}{5-}\;\frac{a^2}{7-} \tag{21}$$

We could also, just for fun, apply the method to a familiar SCF:

$$1 + \frac{1}{1+}\;\frac{1}{1+}\;\frac{1}{1+}\;\frac{1}{1+} \tag{22}$$

Here the differential equation is $y'' = y' + y$. Let $\alpha = (1+\sqrt{5})/2$ denote the golden ratio, and let $\beta = 1/\alpha$. This time the radius of convergence is infinite for both $A(x)$ and $B(x)$. We find, after wading through lots of manipulations, that

$$A(x) = \frac{\alpha e^{\alpha x} + \beta e^{-\beta x}}{\sqrt{5}}, \qquad B(x) = \frac{e^{\alpha x} - e^{-\beta x}}{\sqrt{5}},$$

so the SCF converges to $\lim_{x\to\infty} A(x)/B(x) = \alpha$. (This could have been obtained much more simply!) More generally, we have the following result which is true for any positive real number a:

$$\frac{1+\sqrt{1+4a}}{2} = 1 + \frac{a}{1+}\;\frac{a}{1+}\frac{a}{1+}\frac{a}{1+} \tag{23}$$

The SCF for e

From (15) we obtain, using $a = 1/2$ and repeatedly carrying the '2' into the denominator,

$$\frac{e-1}{e+1} = \frac{1}{2+}\,\frac{1}{6+}\,\frac{1}{10+}\,\frac{1}{14+} \tag{24}$$

and therefore

$$e = 1 + \cfrac{2}{1 + \frac{1}{6+}\,\frac{1}{10+}\,\frac{1}{14+}\cdots}. \tag{25}$$

This is not a SCF. However a SCF may be obtained from (25) using simple algebra, via the following two easily verified identities:

$$\cfrac{2}{2k + \cfrac{1}{a+x}} = \cfrac{1}{k + \cfrac{1}{2(a+x)}}, \tag{26}$$

$$\cfrac{2}{(2k+1) + \cfrac{1}{a+x}} = \cfrac{1}{k + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{(a-1)+x}}}}. \tag{27}$$

Using (26) and (27) repeatedly, we obtain Euler's elegant and very famous result,

$$\begin{aligned} e &= 2 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+} \\ &= [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, \quad \overline{1, 2k, 1}, \quad .]. \end{aligned} \tag{28}$$

Number Crunching

It is interesting to see how close are the approximations to e provided by (24). The GCF gives the following convergents to $(e-1)/(e+1)$:

$$\frac{6}{13}, \frac{61}{132}, \frac{860}{1861}, \frac{15541}{33630}, \frac{342762}{741721}, \frac{8927353}{19318376},$$

Therefore the following are rational approximations for e:

$$\frac{19}{7}, \frac{193}{71}, \frac{2721}{1001}, \frac{49171}{18089}, \frac{1084483}{398959}, \frac{28245729}{10391023}, \quad (29)$$

Comparing these numbers with e, we find that the errors are roughly as follows:

$4\times10^{-3}, -3\times10^{-5}, 1\times10^{-7}, -3\times10^{-10}, 5\times10^{-13}, -6\times10^{-16},$

An impressive convergence rate! – the error in the 10th convergent is less than 3×10^{-25}.

Postscript

The reader may wonder why we have not attempted to compute the SCF

$$[0; 1, 2, 3, 4, \ldots] = \frac{1}{1+}\frac{1}{2+}\frac{1}{3+}\frac{1}{4+}$$

which is surely much more 'natural looking' than $[0; 1, 3, 5, \ldots]$. The answer is a prosaic one. The recursion equations obtained for this SCF are $a_n = na_{n-1} + a_{n-2}$ and $b_n = nb_{n-1} + b_{n-2}$, and the differential equation obtained as a result is:

$$(1-x)y'' = 2y' + y.$$

Unfortunately this equation does not seem to be solvable in closed form; at any rate I am unable to find such a solution! And of course this very effectively puts an end to the investigation

Suggested Reading

[1] Hall and Knight, *Higher Algebra,* Macmillan & Co. Ltd., London, 1960.

[2] Hardy and Wright, *Introduction to the theory of numbers*, Oxford University Press, 1960.

[3] Polya and Szego, *Problems and theorems from analysis*, Problem 85), Springer -Verlag , Vol. I, 1972.

[4] S Barnard and J M Child, *Higher Algebra,* Macmillan and Company, London.

[5] C D Olds, The Simple Continued Fraction Expansion of *e*, in *The Chauvenet Papers*, Vol 2 (MAA).

[6] *Mathematics Magazine,* Problem 1254,Vol. 61, No. 1 (February 1988).

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