Your Vision with and without Trigonometry

Trickeries of a Mundane Pendulum and the Sky-borne Moon

S R Madhu Rao

When you take a look at anything close-by, your brain computes the position of the object you see by solving a lanky triangle – namely the one formed with the object and your two eyes as its vertices. The brain may call other clues also into service while figuring out positions, especially when the lanky triangles prove far too lanky to afford trigonometric solutions with reasonable accuracy. These non-trigonometric alternatives can run into occasional pitfalls, though. Yet even the trigonometric highway isn't always foolproof, either. We shall see in what follows that the resulting consequences can be as instructive as they are amusing.

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Have you ever watched your friend's eyes as they follow a flower or something you hold before them in your hand and slowly move across to-and-fro? If you have, you can't have missed the way their lines of sight swing about in synchrony with your hand's movements. Your friend involuntarily keeps rolling the pair of his/her eyes so that the line of sight of each eye is always directed towards the object in your hand.

The two eyes B and C bound the base of a special triangle of which the flower A in your hand is the far vertex. (Where we need to be more specific, we let B and C designate the so-called nodal points of the eyes' refractive media rather than stand vaguely for the whole of the eyes. An eye's refractive medium includes its cornea and the aqueous and vitreous humor fluids besides the crystalline lens.) The base $a = BC$ of this triangle has a fixed length, about 6 cm, and its other two sides $b = CA$ and $c = BA$ are of course just the lines of sight or to be technically more correct the visual axes of your friend's eyes. Rolling the eyeballs around to get b and c to intersect precisely at A takes some muscular

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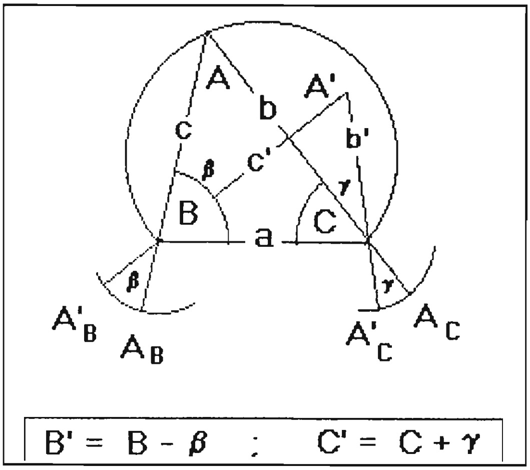
effort. The specific configuration of the eyeballs which this effort brings about tells your friend's brain exactly how wide the angles $B = \angle CBA$ and $C = \angle BCA$ are in the triangle under discussion. It is now clearly possible to determine the triangle's unknown sides b and c via the so-called sine-rule

$$a/\sin(B + C) = b/\sin B = c/\sin C \qquad (1)$$

of elementary trigonometry. Your friend's brain does have an in-built algorithm to carry out an equivalent computation subconsciously, and it is this facility that enables it to sense the presence of A, the flower, unmistakably in a definite position 'out there in the space ahead' – rather than, say, somewhere inside the eyes themselves.

In order to determine when exactly the visual axes of its owner's eyes are both trained sharply on the flower A, your friend's brain first moves his/her eyeballs so as to bring A's retinal images A_B and A_C into two little pits of highest light-sensitivity called the foveae. Then it fine-tunes the eyeballs' orientations so that A_B and A_C are ultimately fused into a single visual entity in its perception. This fusion signals the placing of A_B and A_C almost pixel-by-pixel at precisely matching locations on the two retinas. (See *Box* 1 for some details.) Accordingly, when the fusion occurs, the centers of A_B and A_C perforce coincide with the centers of the foveae. The visual axes A_BB and A_CC joining these centers to the respective nodes of the two eyes must meet at the site of the original object A when extended outwards now (*Figure* 1).

Figure 1.



Peripheral Objects and their Instant Triangulation

It may well happen that quite a few other objects besides A have also cast their images on your friend's retinas. Consider one such additional object A' with its images A'_B and A'_C. Chances are that A'_B and A'_C have formed at disparate, unmatching retinal sites – quite often even out-

Fusion of Retinal Images and the Horopter

Images of one and the same object forming on your two retinas are always sensed initially as two distinct visual entities. Hold the forefingers of your left and right hands one behind the other a little way apart. And hold them directly in front of your eyes at some convenient distance away. You will notice that when you gaze on the closer finger, the farther one looks double, and vice versa.

Your brain can fuse the corresponding images on your two retinas by rolling your eyeballs so as to bring these images pixel by pixel into what are supposed to be matching locations on the pair of retinas. Thus, in *Figure* 1, the images A_B and A_C of the principal object A are at the centers of the two retinas' foveae (which are matching locations), and therefore these appear properly fused. In contrast, the images A'_B and A'_C of the peripheral object A' usually form at unmatching locations, and this results in A' generally being seen double.

The question arises as to whether the images of both A and A' can sometimes co-fuse (each image-pair fusing separately, of course), and if so, under what conditions they might do so. A popular surmise in this connection is that A'_B and A'_C may be expected to fuse together if these images are equidistant from the already fused images AB and AC respectively – that is, if the angles β and γ are equal in *Figure* 1.

The locus of all peripherals A' whose retinal images co-fuse with those of A is known as a horopter. If the popular surmise mentioned above is correct, this horopter will have to be a circle – the circumcircle of our triangle ABC, in fact. (Can you figure out why?)

It looks like stumbling on some really cute clipping of school geometry at this point. In reality, however, the surmise is only approximately valid, and that too for objects in the immediate neighborhood of an onlooker's eyes. For more distant objects, unfortunately, the picture of the horopter as an elegant circle proves to be widely off the mark.

side the foveae. This generally precludes their visual fusion, so much so the object A' is quite often going to be seen double actually. (Such will be the case, at any rate, until the onlooker shifts gaze from A towards A' in surprise. This makes A' the central object and A a mere peripheral. It will then be the turn of A rather than A' to start looking double!). Yet, A' being only a peripheral object, its unfused appearance would not be a cause for much concern anyway. The vital question is whether the brain can now assess the spatial positions of all such sundries as A' in a flash. Can it do so without having to laboriously reorient the eyeballs towards the new objects one by one all over again?

It certainly can. For one thing, the very positions of the four

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images A_B, A'_B, A_C and A'_C on the retinas hold complete information on their pair-wise angular separations

$$\beta = \angle A'_B B A_B \text{ and } \gamma = \angle A'_C C A_C. \qquad (2)$$

For another, the angles $B' = \angle CBA'$ and $C' = \angle BCA'$ of the new triangle $A'BC$ are related to the angles $B = \angle CBA$ and $C = \angle BCA$ of the old one ABC by simply

$$B' = B - \beta \text{ and } C' = C + \gamma. \qquad (3)$$

(To alleviate tedious complications, we have in this context assumed all the eight points $A, B, C, A', AB, A'B, AC$ and $A'C$ to lie in a single horizontal plane.) Thanks to (2) and (3), the unknown sides $b' = CA'$ and $c' = BA'$ of the new triangle $A'BC$, too, can be determined by just the sine-rule

$$a/\sin(B'+C') = b'/\sin B' = c'/\sin C' \qquad (4)$$

These sides b' and c' cannot be termed visual axes in a strict technical sense now, but the loss of nomenclature does not detract in any way from the basic validity of elementary trigonometry.

It thus appears that the brain employs a two-fold strategy to sense spatial depths. It first works out the position of some central object A using a procedure akin to (1) after accurately setting up the eyeballs with orientations appropriate for the task. This is doubtless a relatively slow process, but later on the brain resorts to a fast algorithm on the lines of (2), (3) and (4) to compute the locations of a large number of peripheral objects A' all concurrently. Only this swifter component of the brain's strategy can explain how even a flash view of a real three-dimensional scene (or an equivalent hologram) often successfully evokes a high degree of genuine depth perception in most individuals.

Still, there can be occasions when the available visual cues are highly conflicting, and the brain feels reluctant to accept its own rapid reckonings. Wishing to be doubly sure, it might then prefer to plod along the slower route most of the time, even where only peripherals are involved. Be warned, therefore, that you will not find the brain in quite a hurry to swallow any and every trickery you might care to peddle before it in your experimental projects.

Pulfrich's Deceptive Pendulum

We recommend that you contemplate the trigonometric basis of vision in a truly philosophical vein. When you do so, you will soon realize that at the start every object 'seen' is merely an inference. This is because the data needed for making the relevant computations are not gathered directly from the objects in the world outside, but come from just the images residing on the onlooker's retinas! You can therefore sometimes dupe the brain into inferring things that do not really exist – things that you cannot find, to be precise, at the positions and with the features your brain figures out for them in such illusory contexts.

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One of the most striking examples of such a trickery goes by the name of Pulfrich's pendulum. The experiment is easy enough to be within the reach of even school-children. You just make a simple pendulum with a white-colored playing ball and swing it to-and-fro at close to eye-level in a vertical plane against an unobtrusive dark background. The lighting conditions must be good, of course. Your friend watches the ball's movements with both eyes wide open, but all the time holding a piece of colored cellophane paper (blue or red, and transparent) in front of only the right eye. It then turns out that, as against the strictly plane pendulum that you have taken pains to set up, what your friend sees is a conical one – with the ball seeming to describe an oval-shaped trajectory in a nearly level horizontal plane. Furthermore, the sense of the ball's motion as viewed from above appears to him/her to be unmistakably counter-clockwise!

The explanation? Naturally, it cannot be quite so simple as the demo itself has been. A pensive sidelight is that Pulfrich, who cracked the puzzle correctly, could never experience this illusion himself because he was one-eyed. Now *Figure* 2a/*Figure* 2b illus-

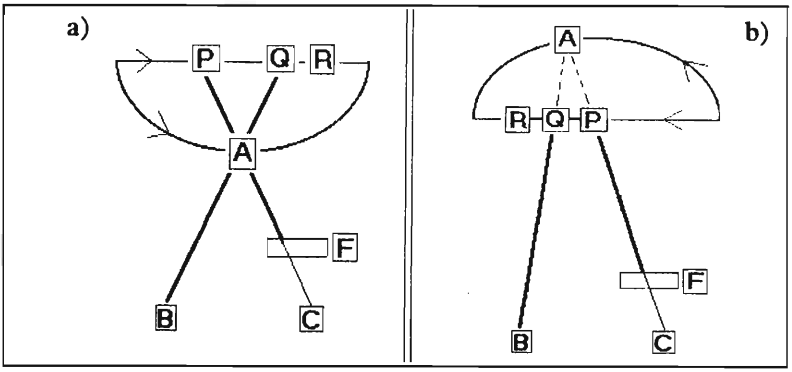

Figure 2.

trates the actual genesis of the illusion. Suppose the pendulum's bob is currently at R. The bob located at R readily casts its images on the onlooker's retinas no doubt, but when can his/her brain begin to sense them actually? Not until a few milliseconds later, since the process depends on the occurrence of certain photochemical reactions and the transmission of nerve impulses. Moreover, the delay is going to be excessively longer in the case of the right retinal image since this one suffers from lack of brightness on account of the intervening cellophane filter F. Working backwards, therefore, you cannot but conclude that what the onlooker's brain is currently sensing is mere history, and history of two different eras, into the bargain. It is sensing the pendulum's left retinal image as it was produced on the left fovea when the bob was at Q, and the right one as produced on the right fovea when the bob was still farther behind – at P! As already elucidated at the outset, the brain's trigonometric algorithms do force it in this situation to perceive a phantom bob at A, the intersection of the corresponding visual axes BQ and CP.

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We conclude our discussion of Pulfrich's illusion with a little poser which our readers should find it easy to resolve: What happens if the cellophane filter is held in front of the left eye instead of the right?

Shadows with the Third Dimension Added in

You are of course familiar with shadows as two-dimensional objects. Can you make them look three-dimensional instead? Surprisingly, the answer is yes, and the experimental design to achieve this result is again well within the reach of school-children.

Set up two incandescent lamps $B^{\sim}$ and $C^{\sim}$ with short filaments some 3 – 6 cm apart in a dark room. (*Figure* 3). Automobile lamps of 12V 20-40W rating housed in suitable enclosures should be a good choice, but other possibilities are not excluded. Use no reflectors. The distance between the lamps should preferably be made variable Pass the light given out by $B^{\sim}$ through an orange filter, and that given out by $C^{\sim}$ through a greenish-blue filter. (You can prepare these filters by combining a layer or two of pink cellophane and yellow cellophane cut up in handy little squares for use with $B^{\sim}$, and similarly of blue cellophane and green cellophane for use with $C^{\sim}$ Let some skeletal object $A^{\sim}$ intercept the filtered rays of light to cast its shadows P and Q on a screen S made of white cloth/paper and placed a meter or so away from the pair of lamps.

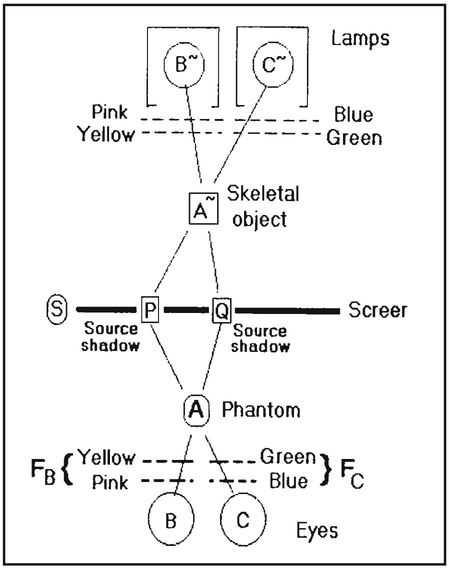

Figure 3.

You now look at the shadows P and Q from the other side of S, but with a specially prepared viewer. The viewer must have an orange filter F_B against the left eye and a greenish-blue filter F_C against the right one. These filters (which again can be fabricated with cellophane) should be so designed that the shadow P totally disappears from view when inspected with F_B, and Q also likewise totally disappears when scanned through F_C. With the viewer donned, therefore, only P's image forms on your right retina, and only Q's on your left retina. This prompts your brain to orient your eyeballs so that these images are moved into matching locations in the respective foveae and fused. You now end up seeing, at the intersection of the visual axes BQ and CP, a single phantom shadow A suspended in space in all its stereoscopic 3-D glory! And what's more, the phantom A would seem to have occultist links with its mundane originator $A^{\sim}$ in a manner redolent of folk-tale fancies. Ask someone to move the tangible A slowly towards the lamps, and you will notice its disembodied form A menacingly approaching your eyes instead. We only hope you will still keep from panicking, right?

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An Age-Old Trick the Moon in the Sky Plays

When it comes to grappling with the far-away moon, the trigonometric route (with a baseline of mere 6 cm separating your eyes) can be of no avail to your brain. Still, functioning in its subconscious mode, it resolutely believes in its primordial intimation that the moon and the stars are in the sky after all, and tries to put two and two together as best it can from this position.

Of what shape is the sky? Perhaps you feel certain that the sky must indeed be a hemispherical dome, for this is what you have been told all along. Your subconscious brain doesn't agree, however. Its conviction is that the sky considered as a segment of a sphere is far less than a hemisphere in size and shape. To make yourself absolutely sure about this amazing truth, just move into a vast plain field with uncluttering landscape around and watch the sky. The blue sky should appear to you then – especially then, that is – as just one-half of a hemisphere or thereabouts.

Why does it appear so? Well, we can only speculate. What you see as the vast blue sky by day is of course just the sun's light scattered by countless molecules and other particles drifting about in the atmosphere. To get a crude picture, therefore, you might think of the sky as a periphery of sorts of the atmosphere surrounding us. It should occasion no surprise at all on this view that the 'height of the zenith *OZ*' should be far less in extent than the 'distance of the horizon *OH*' (*Figure* 4;*O* designates here the position of the observer). But then all our knowledge about the atmosphere, and about the scattering of light taking place in it, is of a highly sophisticated genre. By which route could a naive subconscious brain have ever accessed it at all, in the first place?

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Again we can only speculate. Artists sometimes use a technique called aerial perspective to suggest varying distances. They depict things which are close at hand with a high degree of clarity, but draw remote objects in hazier styles. One of the cues prompting the brain's decision $OZ < OH$ could be just this aerial perspective operating in the sky's context – for do not things near

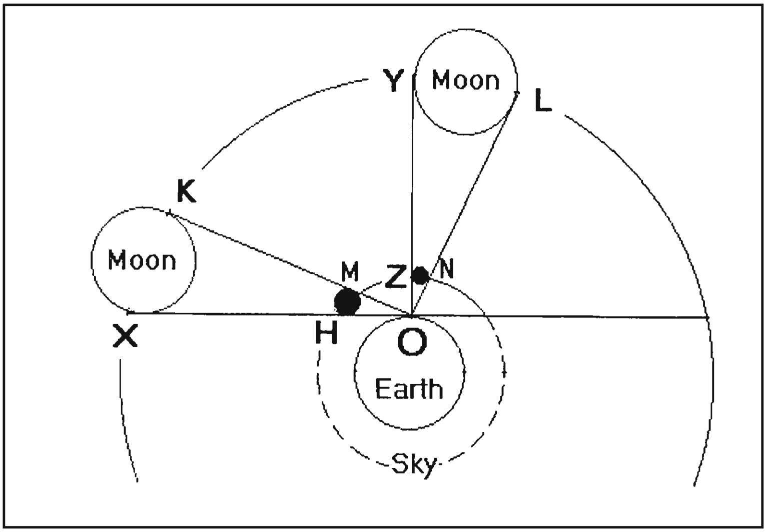

Figure 4.

H, the horizon, generally look hazy? Yet another very crucial factor contributing to the same decision seems to originate from the pattern of clouds seen in different regions of the sky. The average cloud at *Z* casts a vastly bigger image on your retinas than its counterpart near *H*, suggesting that *Z* after all should be a closer site than *H*. These explanations appear plausible at least qualitatively. As for quantitative corroboration, the brain really cannot probe all the way up to the atmospheric limits in the direction of the horizon. It must stop short at the point where a clear vision of clouds and other atmospheric objects becomes impossible due to fog and diminished size.

This results in a much overestimated value of the fraction *OZ*/*OH* as compared to what the atmospheric studies indicate. Yet on the whole the subconscious brain's perception of the sky would appear to be admirably good thus far. It is only in a subsequent step that the brain errs and unwittingly slips into a big illusion.

The illusion sets in when your brain endeavors to paste the moon and the stars on the sky that it has so meticulously conjured up. *Figure* 4 depicts the real rising moon as the disk *XK*, and the real moon overhead likewise as *YL*. Your brain's spontaneous decision, however, is that no moon can ever belong in a station other than the sky! It therefore persuades itself to sense *XK* and *YL* in the form of two phantoms instead. These phantoms are of course the smaller-sized disks *HM* and *ZN* shown blackened in our schematic sketch. Notice how appreciably bigger *HM* is than *ZN*, even though the corresponding source-moons *XK* and *YL* are exactly the same size.

Now you know why the moon (or for that matter any handy little group of stars) looks oversized when near the horizon. Still, truth can be stranger than fiction, and you might feel vaguely skeptical over our blunt reading of the human subconscious brain's curious behavior. It should be more convincing, in any case, to catch this brain in its act under controlled laboratory conditions. And a simulation demo of such a kind is not too difficult to put together, either. All you need do is step back to our 3-D shadows' experiment (*Figure* 3). As you watch the phantom shadow of a fixed skeletal object $A^{\sim}$ in reasonable comfort, ask someone to slowly vary the distance between the lamps $B^{\sim}$ and $C^{\sim}$ You will notice that the phantom grows or shrinks in size according as $B^{\sim}$ and $C^{\sim}$ are moved closer together or farther apart. Do you observe at the same time any correlation between the phantom's changing sizes and its positions in space? What happens, on the other hand, if you take off your viewer and look straight at those real 2-D shadows *P* and *Q* appearing on the screen itself? What you will discover then is that, unlike the phantom they generate, these source-shadows never change much in size at all!

We leave it to our resourceful readers to figure out where the parallel between these observations and the celestial effect comes in. When they do so, their last lingering doubts over our story of the age-old lunar illusion should disappear, and will indeed disappear for good.

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Suggested Reading

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