Fields Medallists

4. Curtis T McMullen

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McMullen has contributed important results in various areas including theory of computation, theory of Teichmueller spaces, three-dimensional geometry and complex dynamics.

In his thesis he proved a fundamental result about algorithms for solving polynomial equations. We know that any polynomial of degree d, with complex coefficients, has d roots. The question is: how do we find the roots? Can we have an algorithm, a procedure to find them which could (say) be written into a program, so that a computer could determine the roots when fed with any polynomial?

A purely iterative algorithm is a prescription associating to each polynomial P of a given degree d a transformation T of the space $\hat{\mathbb{C}}$ of complex numbers together with 'infinity', such that T is given by a formula as a rational function (ratio of polynomials) whose coefficients are given as rational functions in the coefficients of P. Now you start with an initial guess, say z, for the root (which in principle could be any complex number) and successively consider the points z, $T(z)$, $T(T(z))$, ... If T is such that the sequence converges to a root of P this would give us a way of finding roots of polynomials of degree d. It would also be satisfactory if the sequence converges for 'most' (almost all) choices of the initial guess z and for most polynomials of degree d; when this happens we shall say that the algorithm is generally convergent. The well-known Newton's algorithm $T(x) = x - f(x)/f'(x)$, is a purely iterative algorithm for polynomials f of degree d. It is generally convergent for $d = 2$, but not so for higher degrees. It was conjectured by S Smale that there does not exist any generally convergent purely iterative algorithm for polynomials of degrees 4 or more; it may be clarified here that if one allows complex conjugates in the formula for T then by a result due to Shub and Smale there exist root-finding algorithms for all degrees. McMullen in his thesis established the conjecture. On the other hand for cubic polynomials he produced a new generally convergent purely iterative algorithm: for a cubic polynomial in the (reduced) form $P(x) = x^3 + ax + b$, this prescribes applying Newton's algorithm to $f(x) = P(x)/(3ax^2 + 9bx - a^2)$ (and of course its solutions are solutions of $P(x)$).

In another subsequent work McMullen considered, in collaboration with P Doyle, a modified approach involving towers of purely iterative algorithms. This incorporates features of the purely iterative algorithms on the one hand and the idea of solving polynomial equations by radicals (Galois theory) on the other hand. It was shown that such a modified procedure can be applied success-

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Note to the Student Reader

In this four-part article, you will find accounts of the work of the four Fields medallists of 1998. A characteristic feature of mathematics is that research level ideas are often quite difficult to describe in simple terms, owing to the many-runged nature of the concepts involved – abstractions built upon abstractions. Indeed, the words themselves are frequently a source of difficulty, as there are so many technical terms involved. Nevertheless it is necessary that students make at least an attempt to read expository articles which describe current research. This remark is of particular relevance to these articles, as the work described has been judged by the mathematical community to be of far-reaching importance. We urge you to make the effort, and to read the articles in their entirety. In each case, the first one-third or so of the article gives an overview of the topic and highlights the main problems of interest, while the remainder of the article is much more technical and specific. Therefore, if you find yourself in difficulties over the pieces, do not get discouraged!

Editors

fully for polynomials of degree up to 5; it turns out that beyond degree 5 even the tower procedure cannot work.

While the above-mentioned results constitute a major contribution in the theory of computation, the main theme of the work is complex dynamics. Each T as above is a 'dynamical system' (T prescribes the next step from each point which we can iterate) and one can consider its attracting periodic cycles (periodic orbits whose points 'attract' nearby points under p-fold application of the transformation, p being the period of the orbit). Further, the transformations T as above form a family parametrized by polynomials of degree d. In general one considers families of rational maps of $\hat{\mathbb{C}}$ parametrized by algebraic varieties. Such a family is said to be 'stable' if there is a uniform bound on the periods of attracting periodic cycles. The family associated to a generally convergent purely iterative algorithm is stable, since if an attracting periodic cycle has more than one point the iterative sequence z, $T(z)$, ... will keep shuttling between neighbourhoods of different periodic points, and hence will not converge. McMullen classified stable algebraic families of rational maps, drawing upon the work of Mané, Sad and Sullivan on stable families of rational maps and Thurston's work on the so-called critically finite branched coverings of the two-dimensional sphere. From the classification he deduced that for $d \geq 4$ none of the families correspond to a generally convergent purely iterative algorithm, thus proving Smale's conjecture as above. In another related work McMullen also clarified why root-finding algorithms fail to exist in degrees greater than 3, from a topological viewpoint.

McMullen also applied complex dynamics in the study of 3-dimensional manifolds, namely 3-dimensional analogues of surfaces. Thanks to a classification due to Poincaré all surfaces (in an abstract sense) can be realized as joins geometric classification of all 3-dimensional manifolds in an analogous manner was undertaken by W Thurston. He developed a beautiful and intricate theory of 3-manifolds: for a wide class of 3-manifolds which can be built up by gluing finitely many 3-balls along their boundaries in a certain way, known as Haken manifolds, Thurston showed that they can be decomposed into pieces each of which supports one of the eight well-known geometries in dimension 3. The result, known as Thurston's uniformization theorem, is fundamental to contemporary 3-dimensional geometry; however many parts of his theory were not available in final published form and were understood only by experts, in a piecemeal fashion. McMullen has made major contributions to the theory both in terms of giving more palatable proofs for various important results and proving new results about the structure of the manifolds, under various conditions. He proved a conjecture of I Kra that the norm of the so-called Poincaré Θ-operator is strictly less than 1. He also applied the methods of this work to give an analytic proof of Thurston's theorem that any 'atoroidal' Haken manifold is homeomorphic to a hyperbolic manifold, that is, a complete Riemannian manifold with constant curvature -1.

On a closed surface S with at least two handles there exist many 'different' (conformally inequivalent) complex structures and the totality of such structures forms a complex manifold, known as the Teichmueller space of S. L Bers introduced a way of associating to each pair of complex structures on S, a discrete faithful representation of the fundamental group of S into the group of Moebius transformations. Fixing a complex structure as a reference point one obtains from this an embedding of the Teichmueller space, known as Bers' slice, into the space of conjugacy classes of such representations. One of the aims in the theory is to understand the boundary of Bers' slice. McMullen developed powerful techniques for studying the boundary, and proved a conjecture of Bers that the Bers' boundary contains a dense set of 'cusps'; by definition a cusp is a representation which associates to some (hyperbolic) element of the fundamental group a 'parabolic' Moebius transformation.

Unlike in the case of 2-dimensional hyperbolic manifolds, in dimension 3, the celebrated Mostow rigidity theorem says that two compact (or finite volume) hyperbolic 3-manifolds are isometric whenever they are homeomorphic. McMullen generalized this phenomenon by proving a 'geometric inflexibility' or 'asymptotic isometry' theorem for hyperbolic 3-manifolds of infinite volume. This result leads to a simpler approach to Thurston's theorem that if S is a closed surface, ϕ a diffeomorphism of S, and M a 3-manifold obtained by identifying the boundaries of $S \times [0, 1]$ via ϕ then M is homeomorphic to a hyperbolic manifold provided ϕ is a 'pseudo-Anosov' diffeomorphism.

McMullen has also contributed important ideas and deeper insights into renormalization techniques for the study of families of rational maps and their iterations. Consider the family of maps $T(z) = z^2 + c$ of the complex plane, parametrized by complex numbers c. The famous Mandelbrot set consists of those c for which the iterates of $z = 0$ under T are bounded. It is a fundamental open conjecture that for all c in the interior of the Mandelbrot set, the iterates of 0 under T converge to an attracting periodic cycle of T. McMullen proved the conjecture for real values of c, by complementing nicely an important result of J-C Yoccoz in the theory.

Yet another important theme in McMullen's work involves bringing out deep parallels between the theory of Kleinian groups and iterations of rational maps. We may note that Kleinian groups are naturally realized as fundamental groups of hyperbolic 3-manifolds and they can be studied using tools of hyperbolic geometry. A 'dictionary' between the two areas (or correspondence between notions and phenomena observed in the two theories) was introduced by D Sullivan when he solved the 'no wandering domains' problem of Fatou and Julia, for rational maps. The dictionary has been greatly expanded in its depth and scope by McMullen. For example, the geometric inflexibility theorem mentioned earlier has found its analogue in dynamic inflexibility theorem, which has become a powerful tool in applications of renormalization techniques to classification of quadratic-like infinitely renormalizable interval maps, classification of critical circle maps, study of Siegel disks, etc. At a deeper level his work aims to understand the link between chaos, rigidity and universal constants in dynamics.

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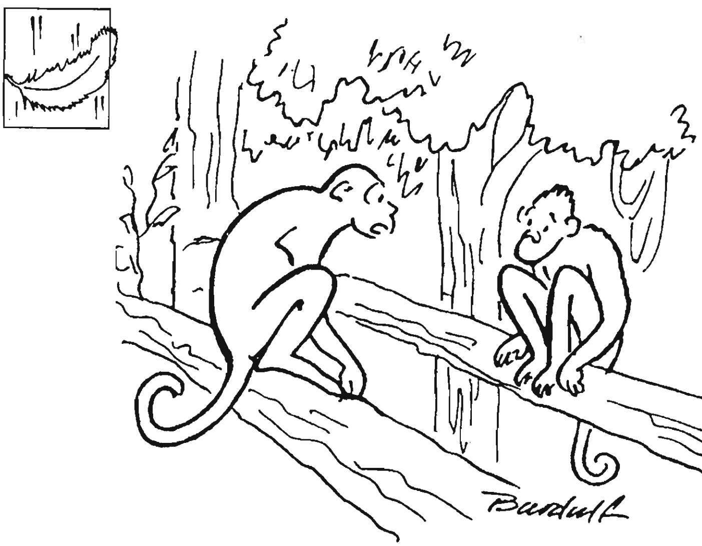

"... Of course, I'd like to evolve into something better, but what could that possibly be?"

From: *Gene Antics*