The Jordan Curve Theorem

2. Conclusions

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Introduction

In the first part of the article (*Resonance*, Vol. 4, No.9) we proved the *Jordan separation theorem* which says that a simple closed curve in E^2 separates it into at least two components. In this concluding part after some preliminary results we will prove the *nonseparation theorem* which states that an arc does not separate E^2 and then put together all the pieces to conclude the proof of the *Jordan Curve theorem*.

Preliminary Lemmas

We had defined when an arc is said to cross a circle. We broaden the definition of crossing as follows:

Definition: Suppose Γ is a piece-wise circular simple closed curve and γ is a piece-wise circular arc. Suppose we have the following situation: There exist $p_1, p_2, p_3, p_4 \in \gamma$ such that $\gamma_{[p,p_2]}$ lies outside Γ and $\gamma_{[p_3 p_4]}$ lies inside Γ while $\gamma_{[p_2,p_3]}$ is a subset of Γ (recall that JCT is true for Γ). Then we consider $\gamma_{[p_2,p_3]}$ as a crossing (see *Figure* 1).

It could happen that $p_2 = p_3$ in which case we are in the context of the earlier definition of crossing.

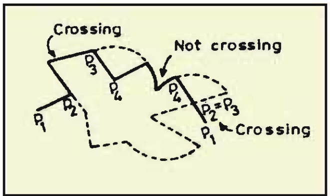
Lemma 2.1. Let γ, Γ be as above and $p, q \in \gamma$ such that p lies inside Γ and q lies outside Γ; then there is at least one crossing between p and q.

Proof. Suppose $\gamma : [0,1] \to E^2$, the map corresponding to γ. Let $\gamma^{-1}(p) = r$ and $\gamma^{-1}(q) = s$; without loss, assume $r < s$. Let

$$\alpha = \sup\{x \leq s | \gamma(x) \in \text{ inside } (\Gamma)\};$$

then obviously $\gamma(\alpha) \in \Gamma$ and $\gamma(\alpha - \epsilon)$ lies inside of Γ for

Figure 1.



small $\epsilon > 0$. Also, as q lies outside (Γ) there exists δ such that $\gamma(x) \in \Gamma$ for $\alpha \leq x \leq \delta$ and $\gamma(\delta + \epsilon) \in$ outside (Γ) for small $\epsilon > 0$. Hence $\gamma[\alpha, \delta]$ is a crossing.

So to 'move' from one component of $(E^2 - \Gamma)$ to the other we have to cross Γ somewhere in between, and conversely, whenever we cross Γ we arrive at the other component. Hence

Lemma 2.2. Let Γ be a piece-wise circular, simple, closed, curve, γ a piece-wise circular arc and a, b, the endpoints of γ. If γ crosses Γ an odd number of times then a and b are in different components of $(E^2 - \Gamma)$.

Suppose γ is a polygonal arc and C is a circle, such that γ crosses C at least once and the end points, a and b, of γ lie 'outside' C. Let γ_C and γ_C^* be the *exterior paths* of γ w.r.t. C. (In the Part 1 of this article we had defined γ_C and γ_C^* when γ is a polygonal line. However, the same prescription works in the case when γ is a piece-wise circular arc.) Let $\delta_C := (\gamma_C - \gamma)$ and $\delta_C^* := (\gamma_C^* - \gamma)$, so δ_C and δ_C^* are subsets of C. In fact δ_C and δ_C^* are finite disjoint union of open arcs of C. Let $I_1, \quad, I_n$ be the sequence in which the (open) arcs are visited while moving from $\bar{a}$ to $\bar{b}$; then $\delta_C = \sqcup_{k=1}^{n} I_k$. Similarly, $\delta_C^* = \sqcup_{l=1}^{m} J_l$.

Lemma 2.3. Let $p, q \in \delta_C$ and α be one of the arcs of C joining p and q. Then α contains an even number of crossings.

Proof. First note that as a, b lie outside C, the total number of crossings in C is even. If $p, q \in I_k$ for some k, then number of crossings is zero in one of the arcs of C and hence the result is true in this case.

Now consider $p \in I_k, q \in I_{k+1}$. If I_k and I_{k+1} are separated by a point of touching (which is not a crossing) then the number of crossings is again zero in one direction.

In the other case, let $\gamma_{[p',q']}$ be the part of γ traveled while moving from I_k to I_{k+1}. Assume $\gamma_{(p',q')} \cap C = \phi$ for the time being. If the direction at p is same as that at q then form the piece-wise circular simple closed curve $\gamma_{[p',q']} \cup p'pqq'$ and call if ξ. Now observe that exactly two situations can arise

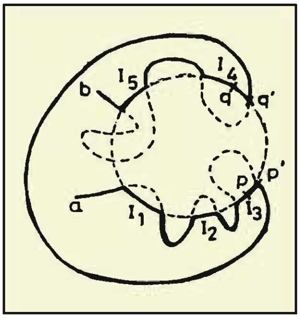


Figure 2.

depending on whether $C - p'pqq'$ lies outside or inside of (ξ):

(i) a, b are both inside (ξ) (*Figure* 2) or

(ii) a, b are both outside (ξ) (*Figure* 3).

Observe that in both cases the number of crossings in $p'pqq'$ is even.

A similar argument proves the lemma if the directions at p and q are different.

Now if $\gamma_{(p'q)} \cap C \neq \phi$, then $\gamma_{[p',q']} \cup p'pqq'$ will be a union of finite number of piece-wise circular simple closed curves. Then we can concentrate on each and every closed curve separately and prove the above claim.

So the statement is true for $p \in I_k$ and $q \in I_{k+1}$. It is easy to see that this implies the general statement! (Why?)

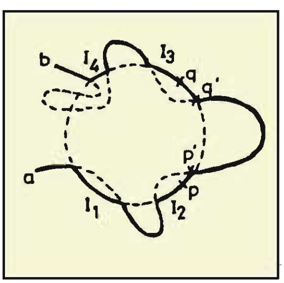
Corollary 1. Suppose $p \in \delta_C, q \in \delta_C^*$ and α is an arc of C joining p and q. Then α contains an odd number of crossings.

Proof. It is enough to consider the case $p \in I_1$ and $q \in J_1$. And in this case $\bar{a}$ (the point where γ hits the circle first) is the only crossing.

Corollary 2. $\delta_C \cup \delta_C^* = \phi$.

A Duality Result: Suppose η and ξ are two piece-wise circular simple closed curves and $p_1, p_2 \in \eta \cap \xi$ such that η 'crosses' ξ at $\eta_{[p_1,p_2]} (= \xi_{[p_1,p_2]})$, then ξ also 'crosses' η at $\xi_{[p_1,p_2]}$. (The notation $\eta_{[p_1,p_2]}$ and $\xi_{[p_1,p_2]}$ are ambiguous, because in each case there are two distinct arcs joining p_1 and p_2. But here we take the arc common to η and ξ i.e. $\eta_{[p_1 p_2]} = \xi_{[p_1 p_2]}$. Also we have not defined crossing in this case; but the definition is obvious.)

Figure 3.



Proof. To prove duality we take sufficiently 'thin' strip neighbourhoods of η and ξ respectively and argue.

Now we state and prove the main lemma of this section.

Lemma 2.4. Let η be an arc joining $p \in \delta_C$, $q \in \delta_C^*$ and satisfying $\eta \cap \gamma = \phi$, then diam $(\eta) \geq \min\{d(a, C), d(b, C)\}$.

Proof. Let α be an arc of C joining p and q. Let $R_1, R_2, \ldots, R_{2s+1}$ be the crossings (of γ and C) on α. Let C_a, C_b be the circles passing through a, b, respectively, concentric with C. We define a relation '$\rightarrow$' on the set $\{R_i\}_{i=1}^{2s+1}$ by $R_i \rightarrow R_j$ if $\gamma_{[R_i, R_j]}$ does not interest C_b and C_a. Without loss, assume radius $(C_b) \leq$ radius (C_a). It is easy to prove that '$\rightarrow$' is an equivalence relation. It partitions the set $\{R_1, \ldots, R_{2s+1}\}$ into equivalence classes. At least one of these equivalence classes has an odd number of points. Let $\{R_{i_1}, \ldots, R_{i_{2\lambda+1}}\}$ be one of them. Let $E, F \in C_b$ such that $R_{i_1}, \ldots, R_{i_{2\lambda+1}} \in \gamma_{(E,F)}$ and $\gamma_{(E,F)} \cap C_b = \phi$. Form $\xi = \gamma_{[E,F]} \cup EF$ (EF is any one of the arcs of C_b joining E and F.) Now α crosses it an odd number of times, hence p and q lie in different components. That η is an arc joining p and q implies $\xi \cup \eta \neq \phi$ but

$$\eta \cap \gamma = \phi \Rightarrow \eta \cup EF \neq \phi \Rightarrow \eta \cup C_b \neq \phi.$$

Also $\{p, q\} \subset \eta \cup C$, hence

$$\text{diam}(\eta) \geq (\text{radius } C_b - \text{radius } C)$$

$= \min\{d(a, C), d(b, C)\}$.

The Nonseparation Theorem

Suppose $\gamma : [0, 1] \hookrightarrow E^2$ is an arc and a, b are two points in $(E^2 - \gamma)$. We will prove that there exists an arc joining a and b, not intersecting γ. As the proof is a bit technical, we will first present the basic plan:

Basic Plan: First we cover up γ with finite number of discs, this gives us a finite collection of circles. Suppose η in any piece-wise circular arc joining a and b, if it intersects γ then we can associate a number $\mu(\eta, \gamma)$ with it,

$$\mu(\eta, \gamma) := \sup\{x | \gamma(x) \in \eta \cap \gamma\} \in [0, 1]$$

Now we choose a suitable circle C from the collection and deform η to η_C such that either η_C does not intersect γ or

$$\mu(\eta_C, \gamma) < \mu(\eta, \gamma) - \delta$$

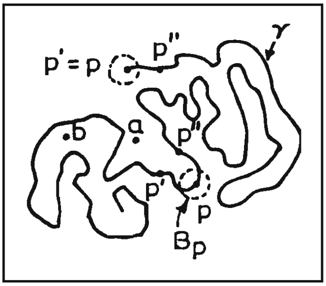


Figure 4.

where δ is a fixed positive number. Then we choose a subarc of η_C joining a and b, and continue the process. As δ is a fixed positive number, after finite number of deformations we will get a piece-wise circular arc not intersecting γ, but joining a and b.

Now we give the proof in detail,

Proof. Let $\alpha := \min\{d(a,\gamma), d(b,\gamma)\} > 0$. As γ is a homeomorphism, for each $p \in \gamma$ there exists $p', p'' \ni p \in \gamma_{(p',p'')}$ and diam $(\gamma_{[p',p'']}) < \alpha/2$. (Of course, at the end points we have to take $p = p'$ and $p \neq p''$.) As E^2 is regular, for each $p \in \gamma$ there exists an open disc $B(p, r_p)$ not intersecting the closed set $\gamma_{[o,p']} \cup \gamma_{[p'',1]} \cup \{a, b\}$. (We denote $\gamma(0)$ by 0 and $\gamma(1)$ by 1.) Again at the end points we have to make suitable modification, for example at $\gamma(0), B(p, r_p)$ must avoid $\gamma_{[p'',1]} \cup \{a, b\}$ only. Let C_p denote the boundary of $B(p, r_p)$. Now $\Omega = \{B(p, r_{p/2}) : p \in \gamma\}$ is an open cover of γ. As γ is compact there exists a finite sub cover

$$S = \{B(q_i, r_{qi/2}) : i = 1, 2, \quad , t\}, \quad \text{let} \quad e := \min\{r_{qi}\} > 0.$$

Now as γ is uniformly continuous, there exists $\delta > 0$ such that

$$d(x,y) < \delta \Rightarrow d(\gamma(x), \gamma(y)) < e/2 \qquad \forall \; x, y \in [0,1]$$

i.e. $d(p,q) > e/2 \Rightarrow d(\gamma^{-1}(p), \gamma^{-1}(p)) > \delta \quad \forall \; p, q, \in \gamma\,.$ $\qquad (*)$

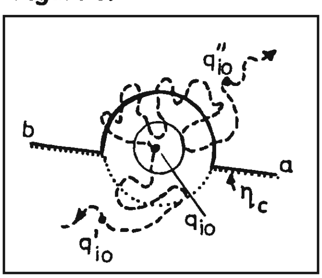
Let η be a piece-wise circular arc joining a and b, suppose $\eta \cap \gamma \neq \phi$, then there exists $i_0 \in \{1, \quad , t\}$ such that $\gamma(\mu(\eta,\gamma)) \in B(q_{i_0}, \gamma_{q_{i0/2}})$.

Let C denote the boundary of $B(q_{i_0}, r_{q_{i_0}})$. Consider η_C and η_C^*; $\gamma_{[q''_{i_0},1]}$ does not intersect them (as it does not intersect c and η) and as diameter of the part of γ from $\gamma(\mu(\eta,\gamma))$ to q''_{i_0} is less than diam $\gamma_{[q'_{i_0}, q''_{i_0}]} < \alpha/2$, so that part does not intersect one of η_C and η_C^* (Lemma 2.4), say η_C. So if η_C intersects γ then $\mu(\eta_C,\gamma) < \mu(\eta,\gamma)$. Moreover $\gamma(\mu(\eta,\gamma))$ lies inside $B(q_{i_0}, r_{q_{i0/2}})$ and η_C lies outside $B(q_{i_0}, r_{q_{i_0}})$ hence

$$d(\gamma(\mu(\eta,\gamma)), \eta_C) > r_{q_{i_{0/2}}} \geq e$$

So using $(*)$ we conclude $\mu(\eta_C, \gamma) < \mu(\eta,\gamma) - \delta$.

Figure 5.



Proof of Jordan Curve Theorem

Let Γ be a simple closed curve in E^2 and $\{\Gamma_\alpha\}_{\alpha\in A}$ be the components of $(E^2 - \Gamma)$. Now as Γ is topologically closed, each Γ_α is open. Openness of Γ_α assures us that A is a countable set.

Lemma 4.1 (i) $Bd\ \Gamma_\alpha \subset \Gamma$ for all $\alpha \in A$;(ii) exactly one of Γ_αs has bounded complement.

Proof. (i) If possible choose $p \in (Bd\ \Gamma_\alpha - \Gamma)$. Now as Γ_α is open $p \notin \Gamma_\alpha$. Therefore there exists $\beta \neq \alpha$ such that $p \in \Gamma_\beta$. But this implies $\Gamma_\alpha \cap \Gamma_\beta \neq \phi$ (as Γ_β is open), a contradiction. (ii) Take a circle C such that Γ lies inside (C). Then $\Gamma_\alpha \cap I(C) \neq \phi$ for all α. Now if $\Gamma_{\alpha_0} \cap O(C) \neq \phi$ for some $\alpha_0 \in A$, then $\Gamma_{\alpha_0} \cap C$ is non-empty. But $C \cap \Gamma = \phi$ implies $C \cap Bd\ \Gamma_{\alpha_0} = \phi$. Hence $C \subset \Gamma_{\alpha_0} \Rightarrow O(C) \subset \Gamma_{\alpha_0}$.

Notation: $O(\Gamma) = \Gamma_{\alpha_0}$ (the unique component with bounded component) and $I(\Gamma) = \cup_{\alpha\neq\alpha_0}\Gamma_\alpha$.

The non-separation theorem implies

Lemma 4.2. $Bd\ \Gamma_\alpha = \Gamma\ \ \forall\ \alpha$.

Proof. Suppose $Bd\ \Gamma_\alpha \neq \Gamma$ for some α. As $Bd\ \Gamma_\alpha$ is a closed set and $Bd\ \Gamma_\alpha \subset \Gamma$ there exists a subarc of Γ say η, such that $Bd\ \Gamma_\alpha \subset \eta$. Now, from the non-separation theorem η does not separate E^2. Take $p \in \Gamma_\alpha$ and $q \in \Gamma_\beta$ $(\beta \neq \alpha)$, there exists an arc ξ (say), in $(E^2 - \eta)$ joining p and q. Since $p \in \xi \cap \Gamma_\alpha$ and $q \in \xi \cap \bar{\Gamma}^c_\alpha$, this implies that $\xi = (\xi \cap \Gamma_\alpha) \cup (\xi \cup \bar{\Gamma}^c_\alpha)$ is a separation of ξ. This contradicts the connectedness of ξ.

Lemma 4.3. Suppose γ and Γ are two simple closed curves in E^2 and $I(\gamma) \neq \phi$. Then γ intersects $O(\Gamma)$ and $I(\Gamma)$ implies that Γ intersects $O(\gamma)$ and $I(\gamma)$.

$$O(\Gamma)\cap\gamma \neq \phi\ \ \&\ \ I(\Gamma)\cap\gamma \neq \phi \Rightarrow \Gamma\cap O(\gamma) \neq \phi\ \ \&\ \ \Gamma\cap I(\gamma) \neq \phi.$$

Let γ_α be a component of $(E^2 - \gamma)$. We know that $Bd\ \gamma_\alpha = \gamma$. Therefore,

$$O(\Gamma)\cap\gamma \neq \phi \Rightarrow O(\Gamma)\cap\gamma_\alpha \neq \phi \Rightarrow \Gamma\cap\gamma_\alpha \neq \phi\ \ (\text{as } \gamma_\alpha \text{ is connected}).$$

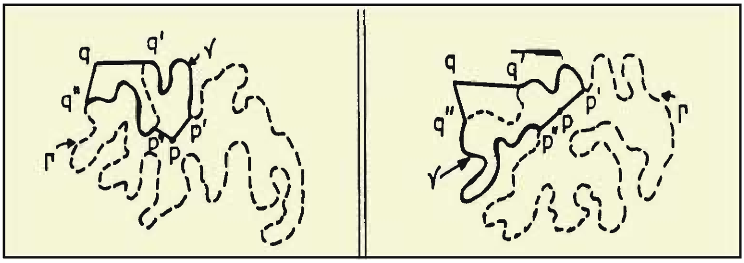


Figure 6 (left).
Figure 7(right).

Therefore, $\Gamma \cap O(\gamma) \neq \phi$. Arguing similarly, $\Gamma \cap I(\gamma) \neq \phi$.

Now that we have all the necessary results, proof of the Jordan Curve Theorem can be quickly disposed off.

From the separation theorem it follows that $I(\Gamma) \neq \phi$ for any simple closed curve Γ We will prove that $I(\Gamma)$ is in fact a component of $(E^2 - \Gamma)$. If not, then there exist two distinct components $\Gamma_\alpha, \Gamma_\beta \subset I(\Gamma)$. Take $p \in \Gamma_\alpha$, we can find p', p'' in Γ such that the line segments $\overline{pp}'$ and $\overline{pp}''$ excluding the points p' and p'' respectively, lie in Γ_α. Similarly for a point $q \in O(\Gamma)$, we can find q', q'' in Γ Let γ be the simple closed curve formed by $\widehat{p'pp''}$, $\widehat{q'qq''}$, part of Γ between p' and q' and part of Γ between p'' and q'', as shown (*Figures* 6 and 7). Then $\gamma \cap O(\Gamma) \neq \phi$ and $\gamma \cap I(\Gamma) \neq \phi$ implies that $\Gamma \cap 0(\gamma) \neq \phi$ and $\Gamma \cap I(\gamma) \neq \phi$. Hence $0(\gamma) \cap \Gamma_\alpha$ and $I(\gamma) \cap \Gamma \neq \phi$ $(Bd\, \Gamma_\beta = \Gamma)$. But $\Gamma_\beta \cap \gamma = \phi$ which contradicts the connectedness of Γ_β. Hence $I(\Gamma)$ is a component of $(E^2 - \Gamma)$.

Thus a simple closed curve in the plane separates it into exactly two components, one bounded and the other unbounded.

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Suggested Reading

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