Think It Over

This section of* Resonance *is meant to raise thought-provoking, interesting, or just plain brain-teasing questions every month, and discuss answers a few months later. Readers are welcome to send in suggestions for such questions, solutions to questions already posed, comments on the solutions discussed in the journal, etc. to* Resonance *Indian Academy of Sciences, Bangalore 560 080, with "Think It Over" written on the cover or card to help us sort the correspondence. Due to limitations of space, it may not be possible to use all the material received. However, the coordinators of this section will try and select items which best illustrate various ideas and concepts, for inclusion in this section.

A Problem from Paul Erdös and N H Anning

Solution to a problem which appeared in Think-It-Over, *Resonance*, Vol.3, No.3, March 1998, p.70.

Answered by: Shailesh A Shirali, Rishi Valley School, AP, India. and Hariharan Narayanan, B-54, IIT, Mumbai 400 076, India.

Show that (a) for any integer $n > 2$ one can find n distinct points in a plane, not all on a line, such that the distances between them are all integers; (b) it is impossible to find infinitely many points, not all on a line, with this property.

Solution to (a). Let C be the circle with centre $(0,0)$ and radius 1, and let ℓ_t be the line with slope t passing through the point $A = (-1,0)$. Then C and ℓ_t meet at A and P_t, where

$$P_t = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) \tag{1}$$

Observe that if t is rational, then P_t is a rational point; that is, its coordinates are rational. (This formula permits us, in principle, to enumerate all rational points on C.) Let B denote the point $(1,0)$. Then $AP_t = 2/\sqrt{1+t^2}$ and $BP_t = 2t/\sqrt{1+t^2}$, so if t is such that t and $\sqrt{1+t^2}$ are rational, then $|AP_t|$ and $|BP_t|$ are both rational. This can be accomplished by letting $t = (1 - r^2)/2r$, where $r \neq 0$ is rational. So we can find infinitely many rational points P_t on $\mathcal{C}$ such that the distances $|AP_t|$, $|BP_t|$ are rational. Next, Ptolemy's theorem ("*If* $ABCD$ *is a cyclic quadrilateral then* $|AB|\ |CD| + |AD|\ |BC| = |AC|\ |BD|$") implies that for any four points on a circle, if five of the six distances determined by pairs of these points are rational, then so is the sixth one. Using this we see that if u and v are non-zero rational numbers such that P_u and P_v (as defined by (1)) are points at rational distances from A and B, then the distance $|P_uP_v|$ is rational. So for any integer $n > 1$ we can find n rational points on $\mathcal{C}$ such that the distances between them are all rational. Scaling up by some suitable integer now allows us to ensure that the distances between the points are all integers.

Erdős and Anning offer another solution. Let m^2 be an odd number with d divisors, where $d > n$, and consider the following equation in integers x_i and y_i:

$$m^2 = x_i^2 - y_i^2.$$

This has d solutions. Now consider the $d + 1$ non-collinear points $(m, 0)$ and $(0, y_i)$ for $i = 1, 2, \ldots d$. It is easy to check that the distances between the points are all integers.

The solution received from Hariharan N is as follows. Let $n > 2$ be a given positive integer. Let p_i denote the i^{th} prime, and let $x = 2p_1p_2 \cdots p_{n-2}$. Let ℓ be a fixed line and let Q be any point on ℓ. Locate points $Q_1, Q_2, \ldots, Q_{n-2}$ on ℓ, all on one side of Q, such that

$$|QQ_i| = \left(\frac{x}{2p_i}\right)^2 - p_i^2 \qquad (i = 1, 2, \ldots, n-2).$$

It is clear that $|QQ_i|$ and $|Q_iQ_j|$ are integral for all i and j. Now let P be located such that $PQ \perp \ell$ and $|PQ| = x$. It is easy to check that

$$|PQ_i| = \left(\frac{x}{2p_i}\right)^2 + p_i^2 \qquad (i = 1, 2, \ldots, n-2).$$

[1] This is not the original proof given by Erdös and Anning.

It follows that $\{P, Q, Q_1, Q_2, \quad , Q_{n-2}\}$ is a set of n points, not all collinear, such that the distance between any two of the points is integral.

Remark. The choice '$p_i = i^{\text{th}}$, prime' is quite unnecessary; we could just as well let $p_i = i$.

Solution to (b). [1] Let A, B, C be three points, not on a line, such that the distances between them are integers. Suppose we wish to find a fourth point P at integral distance from each of A, B, C. The triangle inequality implies that

$$|PA - PB| \leq |AB|, \qquad |PB - PC| \leq |BC|.$$

Let $k = \max(|AB|, |BC|)$. Then $|PA - PB|$ assumes one of the values 0, 1, 2, .. k, and likewise for $|PB - PC|$. So P lies on one of the hyperbolas

$$\{Q : |QA - QB| = i\} \qquad i = 0, 1, 2, \quad , k,$$

and also on one of the hyperbolas

$$\{Q : |QB - QC| = j\}, \qquad j = 0, 1, 2, \quad , k.$$

Each of these families has $k + 1$ members, and as two hyperbolas intersect in at most 4 points, there are at most $4(k+1)^2$ such points P. So the question of finding infinitely many such points P does not arise.

Please Note: *Resonance*, Vol.4, No.8.

On line 2 on page 37: $f(X_1, .., X_n) = \Sigma_i \in T\, g_i(X_1, .., X_n) f_i(X_1, .., X_n)$ should read as $f(X_1, .., X_n) = \Sigma_{f_i \in T}\, g_i(X_1, .., X_n) f_i(X_1, .., X_n)$

Of example 2.3 on page 43: (ii) The only ideals in $\mathbf{Z}$ are principal ideals,.viz., $\mathbf{Z}_m = < m >$ ... should read as (ii) The only ideals in $\mathbf{Z}$ are principal ideals,viz., $\mathbf{Z}m = < m >$ ...