David Hilbert

It will be difficult to find a twentieth century mathematician working in an area that was not touched by David Hilbert. There is Hilbert space, Hilbert scheme, Hilbert polynomial, Hilbert matrix, Hilbert inequality, Hilbert invariant integral, Hilbert norm-residue symbol, Hilbert transform, Hilbert class-field, Hilbert basis theorem, Hilbert irreducibility theorem, Hilbert nullstellensatz.

Hilbert also changed the way mathematicians think about their subject. The axiomatic spirit in which modern mathematics is done owes much to him.

In an address to the International Congress of Mathematicians in 1900, he proposed a list of 23 problems that, in his opinion, should be the principal targets for mathematicians in this century. This famous list, now called Hilbert's Problems, has directed the work of several leading mathematicians.

David Hilbert was born on January 23, 1862 near Königsberg, then the capital of East Prussia, now renamed as Kaliningrad in Russia. The seven bridges on the river Pregel flowing through this town are associated with one of the most famous problems in mathematics. The solution of this problem by Euler became the first theorem in graph theory. The famous philosopher Kant lived here and the great mathematician Jacobi taught at the university of this town.

David's parents were Maria and Otto Hilbert. David's father was a judge. Hilbert's teachers at Königsberg, then a leading university of Germany included H Weber and A Hurwitz. Among his fellow students was H Minkowski. Hilbert, Hurwitz and Minkowski began here a life-long friendship that nourished them in their scientific and personal lives.

Hilbert's research began with the *theory of invariants*, a subject with roots in geometry and number theory. The theory had begun with the work of A Cayley and was developed further by J J Sylvester, R Clebsch and P Gordan. Hilbert changed the face of the subject in two ways. First he broadened the scope of the theory by introducing the notion of invariants for general groups. Second, he proved the existence of a finite basis for the ring of invariants, *not* by explicit computations as others before had done, but by a general *existential* argument. Such an argument, now so commonly used, proceeds by showing that an object must exist, because if it did not, a contradiction would follow.

Gordan, then considered the 'King of Invariants', on seeing Hilbert's proof remarked "This is not Mathematics. It is Theology". It is somewhat ironic that Hilbert got a crucial idea for his theorem on invariants by studying the work of L Kronecker who was a staunch opponent of such non-constructive proofs.

To meet such criticisms, and to show the way out of certain paradoxes that had arisen in the theory of sets, Hilbert advanced the doctrine of *formalism* as opposed to *logicism* of B Russell and *intuitionism* of L E J Brouwer. At issue was the very nature of mathematical proof. Today, most mathematicians have accepted the formalist viewpoint.

Hilbert's work on invariants became the cornerstone of modern algebra. He went on to do equally fundamental work in geometry, number theory, analysis, differential and integral equations, calculus of variations, and mathematical physics. In his *Zahlbericht* (1897), a monumental report written at the invitation of the German Mathematical Society, he presented a unification of the known results on algebraic number fields as "an edifice of rare beauty and harmony". In his book *Grundlagen der Geometrie* (1899) he laid down a list of complete axioms of Euclidean geometry. He examined the logical relations between these axioms and showed their *independence* by constructing *models* in which all but one of the axioms are satisfied. He went on to show that this axiomatic system is as *consistent* as the theory of real numbers.

Hilbert spent most of his professional life at Göttingen, for a long time regarded as the mathematics capital of the world. Among his predecessors here had been C F Gauss and B Riemann; among his contemporaries were F Klein, E Landau, H Weyl and Emmy Noether. The physicist Max Born began his scientific career as Hilbert's assistant. Born's first two assistants, when he later established an institute for physics at Göttingen, were W Pauli and W Heisenberg.

Hilbert's work on integral equations and eigenvalue problems was inspired by the important papers of E Fredholm. Just as he had done in other subjects, Hilbert laid emphasis on the fundamental principles of the subject. This laid the foundation for the theory of Hilbert spaces developed by J von Neumann and others. The classic book *Methods of Mathematical Physics* by Courant and Hilbert was also an outcome of this work. Here, several problems of differential and integral equations were formulated as problems in infinite-dimensional linear algebra. In this book physicists found many mathematical tools they needed to develop the new quantum mechanics. It is most remarkable that the word *spectrum* Hilbert had used to describe some quantities associated with linear operators later turned out to be exactly the spectrum associated with atomic emissions.

The first approach to quantum mechanics was the *matrix mechanics* of Heisenberg, developed further by Born and Jordan. When they approached Hilbert for advice, he replied that he did not know much about matrices except that he had thought of them in connection with some differential equations and perhaps they should look for such equations associated with their matrices. His suggestion was ignored as being a shot in the dark. However, soon E Schrödinger proposed an alternative approach to quantum mechanics called *wave mechanics*. This used differential equations and was very different from *matrix mechanics*. Soon however, the two theories were shown to be equivalent, just as Hilbert had anticipated.

Of course, Hilbert could also be wrong in his judgements. In a lecture in 1919, he gave some examples of problems in number theory that are simple to state but extremely hard to solve. He mentioned the Riemann hypothesis, Fermat's Last Theorem, and the conjecture that $2^{\sqrt{2}}$ is a transcendental number (Hilbert's seventh problem in the famous list). He then added that he might see the proof of the Riemann hypothesis in his life time, that the youngest members of the audience might live to see Fermat's Last Theorem proved, but no one present in the hall would live to see a proof of transcendence of $2^{\sqrt{2}}$. Things did not go the way Hilbert had predicted. The transcendence of $2^{\sqrt{2}}$ was established by A Gel'fond in

1934 when Hilbert was alive; Fermat's Last Theorem was proved by Andrew Wiles in 1994 when perhaps all the members of Hilbert's audience in 1919 were dead; the Riemann hypothesis is yet to be proved. Incidentally, among Hilbert's first works in number theory is a new and simple proof of the transcendence of the number e (first established by Hermite) and of the number π (first established by Lindemann, Hilbert's teacher at Königsberg).

As a person, Hilbert was fair, firm and bold. In 1914, when the German government publicised a declaration in defence of its war actions signed by its most famous scientists, Hilbert's name was missing. The declaration included several statements beginning "It is not true that..." Hilbert refused to sign it on the ground that he could not ascertain whether these statements were true. In 1917 he wrote and published a tribute to the French mathematician G Darboux on his death. This tribute to an 'enemy' outraged some students who demonstrated at Hilbert's home demanding repudiation from him and the destruction of all copies of the publication. Hilbert refused and then insisted on getting an apology from the university. When the conservative professors of the university opposed the appointment of Emmy Noether, a mathematician of the highest calibre, because she was a woman, Hilbert retorted that the University Senate was not a bathhouse where women could not enter. He was outraged by, and was incredulous at, the dismissal of his Jewish colleagues by the Nazis.

He lived to see the tragic destruction of his great centre of mathematics amidst the bigger tragedy of his country. He died on February 14, 1943. The times were such that only about ten persons attended his funeral service, and the news of his death reached the outside world several months later.

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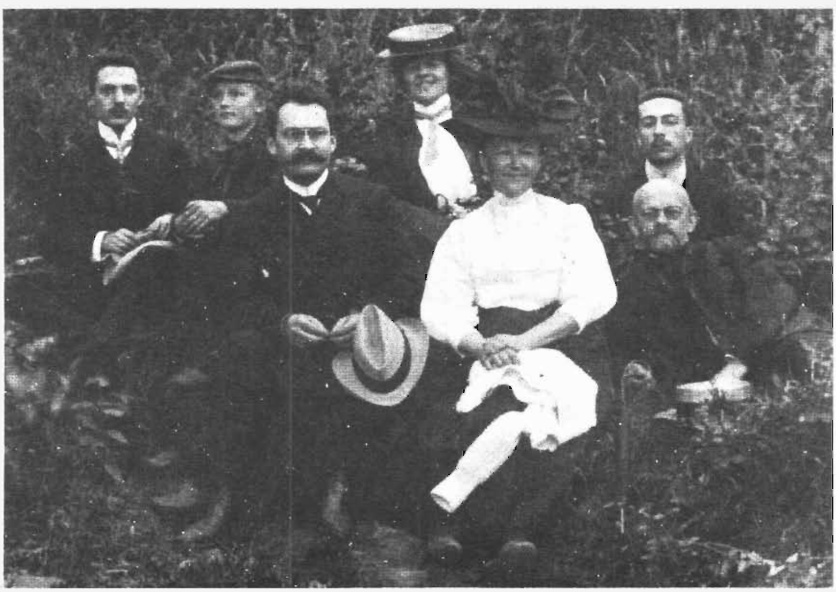

Top: Felix Klein
Bottom:
Alfred Haar,
Franz Hilbert,
Minkowski, ... ,
the Hilberts,
Ernst Hellinger.