The Work of The Fields Medallists: 1998[1]

3. Maxim Kontsevich

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The last two decades have seen a burgeoning interaction between mathematics and theoretical physics, taking place at the very frontiers of the two subjects. Kontsevich is one of the most imaginative of those working this rich seam.

Intersection Theory on $\overline{M}_{g,n}$

Kontsevich first became known to the international mathematical community in the late eighties, as the Russian wünderkind who proved E Witten's conjecture on the generating functional of intersection numbers of stable classes on the (compactified) moduli space of curves.

We consider compact Riemann surfaces X of genus g. To 'build' such an X, one starts with a compact two-dimensional manifold (g is just the number of 'holes') – and one defines a holomorphic structure on it. It turns out that the resulting object has two 'avatars' – first, as a complex algebraic curve, the set of simultaneous zeroes of finitely many polynomial equations with complex coefficients, and secondly, as the quotient of the complex upper-half-plane by a discrete group of isometries of the hyperbolic metric. As the equations change, or equivalently, as the discrete group deforms, so does the Riemann surface. The space that parametrizes these deformations is a geometric object of great mathematical significance, the 'moduli space of curves', M_g. The dual description – in algebro-geometric and group-theoretic terms – is part of the fascination of M_g.

String theory – the would-be Theory of Everything – focussed the attention of theoretical physicists on this space. Briefly, string theory is built out of 'conformal field theory' on the 'world-sheet' of the string. That is, a string propagates in space-time, the space-time co-ordinates are quantum fields on the surface traced out by the string, obeying the laws of a conformal field theory. But to define this latter theory, one needs to introduce the structure of a Riemann surface on the world-sheet. To make the resulting theory independent of this auxiliary choice, one integrates over the parameter space of such choices, viz. M_g.

It turns out that one has to consider a surface together with a choice of n labelled points on it. The space that parametrizes these is denoted $M_{g,n}$. Further, one has to allow the surface to have singularities – this results in the replacement of $M_{g,n}$ by its 'compactification' $\overline{M}_{g,n}$.

Mathematicians have long been interested in the topology of $\overline{M}_{g,n}$, in particular in its cohomology ring. As the name implies, the

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Note to the Student Reader

In this four-part article[1], you will find accounts of the work of the four Fields medallists of 1998. A characteristic feature of mathematics is that research level ideas are often quite difficult to describe in simple terms, owing to the many-runged nature of the concepts involved – abstractions built upon abstractions. Indeed, the words themselves are frequently a source of difficulty, as there are so many technical terms involved. Nevertheless it is necessary that students make at least an attempt to read expository articles which describe current research. This remark is of particular relevance to these articles, as the work described has been judged by the mathematical community to be of far-reaching importance. We urge you to make the effort, and to read the articles in their entirety. In each case, the first one-third or so of the article gives an overview of the topic and highlights the main problems of interest, while the remainder of the article is much more technical and specific. Therefore, if you find yourself in difficulties over the pieces, do not get discouraged!

Editors

[1] Part 4 will appear in a subsequent issue.

elements of this ring are topological invariants – cohomology classes – which can be multiplied. The ring structure is summarized in a collection of rational numbers called intersection numbers. Witten's conjecture – motivated by 'two-dimensional quantum gravity' – concerned a generating function for these numbers. I will not define this function – suffice it to say that it is a formal series in infinitely many variables, and it encodes information about $M_{g,n}$ for all g and n. Witten's conjecture was that this function satisfies an infinite hierarchy of equations of the Korteweg–de Vries type.

Kontsevich's proof is an inspired combination of combinatorial and geometric reasoning, involving the asymptotics (as $N \to \infty$) of 'matrix integrals', integrals of suitable $U(N)$-invariant functions over the space of $N \times N$ matrices. Such integrals were very much in the air, having been originally considered in the context of nuclear physics in the fifties, and revived precisely in the context of matrix models of string theory, from where Witten's conjecture arose.

Knot Invariants

The second major contribution of Kontsevich was a reformulation of the knot invariants of Vassiliev in terms of iterated integrals over the knot. A knot is a smooth embedding of the circle in three-space, and a classical problem in topology is to be able to distinguish knots. Anyone who has struggled with a tangle of string knows how hard it is to tell whether there is a genuine 'knottedness' in it or not; V Jones was awarded the Fields Prize in 1990 partly for the discovery of new invariants which could 'distinguish' knots which could earlier not be told apart. (E Witten defined the same invariants as Jones from the point of view of Chern–Simons field theory.) Vassiliev had introduced (via a complicated combinatorial construction) generalizations of the Jones–Witten invariants.

Kontsevich's work grew out of the study of the functional integrals involved in Chern–Simons theory. (Related work was done by Axelrod and Singer.) This involves formal integrals over the (infinite-dimensional) space of nonabelian gauge fields (= connections on principal G-bundles) on three-dimensional manifolds, in particular the complement of a knot. Though the integrals are ill-defined, one can define a formal series (in inverse powers of an integer k, called the level), each term of which involves only the critical points of the integrand – the space of zero field-strength ('flat') gauge fields. This series need not converge, hence the term 'formal'. These integrals are now ordinary integrals, but over non-compact spaces, and depend on auxiliary choices. Kontsevich overcame the non-compactness by constructing a very clever compactification of the relevant integration domains, and then showed the independence from the auxiliary choices by a deft use of homological algebra.

Quantum Cohomology and Mirror Symmetry

Physicists, studying the propagation of strings in Calabi–Yau threefolds, uncovered two very intriguing phenomena. One is that the cohomology ring admits nontrivial ('quantum') deformations. The other is a mysterious ('mirror') symmetry which produces, for every such threefold, another, called its mirror. Kontsevich (in collaboration with Y Manin) made crucial contributions to the understanding of the relation between these two phenomena.

One of the concerns of numerative (algebraic) geometry has been counting the number of curves (of given topological type) in ambient varieties. In a modern setting, this problem re-appeared in the context of Gromov–Witten classes of symplectic manifolds, which are defined in terms of 'pseudo-holomorphic curves', which are the appropriate generalization when the ambient space is no longer an algebraic variety, but a symplectic manifold. Once these are defined – and this is a nontrivial problem to which Kontsevich contributed in an essential way – they can be used to define the quantum cohomology ring of the ambient space. One of his important insights was that the associativity of the quantum multiplication can be understood in terms of a 'Dubrovin prepotential' on the cohomology ring.

Deformation Quantization

What all the above makes clear is the felicity with which Kontsevich combines ideas from theoretical physics with insights from diverse branches of mathematics. In particular, the work shows remarkable combinatorial and algebraic insight. And these strengths are revealed further in his recent solution of the problem of deformation quantization.

The problem is the following. Consider the space of functions on a manifold M. This is a commutative algebra. *A quantum deformation* of this algebra is a formal deformation as a (not necessarily commutative) associative algebra, with the deformation parametrized by a parameter $\hbar$. (The deformation is described as a formal series in $\hbar$.) One checks easily that (up to reparametrization) the leading order in $\hbar$ defines a Poisson bracket on the space of functions. The problem of deformation quantization is the converse – is every Poisson bracket realizable as the first term of a quantum deformation? Special cases had been dealt with before. In 1997 Kontsevich answered the question in the affirmative, using constructions that first occurred in his work on Chern–Simons theory. Even though he himself says 'it is not clear if 'deformation quantization' is natural for quantum mechanics', the ideas involved in the proof are very interesting from many different points of view – homological algebra and arithmetic among them.

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At present, my son and I are extremely interested in genetics and heredity.

From: *Gene Antics*