Music, Mathematics and Bach

2. Patterns in Bach's Music

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In the earlier part of this article, an introduction to musical scales, counterpoint and harmony was given. In this part these themes are developed further with reference to Bach's music and the emergence of the equally tempered scale.

Bach's Music: Counterpoint, Canon, Fugue and other Patterns

Most readers of this journal may not be very familiar with Bach's music, and this section, I hope, will prove a good introduction to what I would consider one of the high points of music, not only of the west, but of the world.

J S Bach (1685–1750) was in his own time more famous as an organist than as a composer. The trend in the baroque age, of which Bach was among the last major figures, was to move away from polyphony and towards a simpler harmonic style; Vivaldi's concertos, over 500 in number, are delightfully easy listening with catchy melodies and unsophisticated accompaniment (Stravinsky allegedly remarked that Vivaldi 'wrote the same concerto 500 times'). Bach instead concentrated on perfecting the apparently dying art of polyphony. Even his simplest and most unassuming works have a contrapuntal element, and his more sophisticated pieces have an intricacy which can be mind-boggling. But this was appreciated only by a few at the time: though many music teachers, and composers such as Beethoven, knew of Bach's work and held it in great regard, the general public became aware of it only in the 1820s, principally through the efforts of Felix Mendelssohn.

As described above, polyphony or counterpoint is the interplay of two or more distinct melodic voices to form a coherent whole.

The melody of the second voice can be entirely different from the first, or it can be similar with variations. An interesting possibility, and one of the oldest, is when the second voice is the same as the first, but starts off after a small time lag: the first voice begins the melody, then as it continues the second voice joins in with the same melody after a few bars. The results with a randomly chosen melody (or 'theme') would be execrable, but if the original theme is carefully written the effect can be striking. This is an example of a 'canon' One can have various variations on this basic canonic form: the second melody can be slower or faster than the original, played backwards, 'inverted' , and so on. To convert these academic exercises into interesting music is far from trivial.

Both systems have a base note or 'tonic'. In Indian music the tonic 'sa' is played throughout by the tanpura, and one never deviates from it, while in Western music the music can start in one key (with a particular tonic – such as C) and then 'modulate' to a different key, with a different tonic.

Another thing one can do with the second melody is to play it in a different key, that is, at a different pitch. This is a good point at which to mention another important respect, apart from the 'vertical component', in which Western and Indian music differ: Western music allows the idea of 'modulation' from one key to another. That is to say, though both systems have a base note or 'tonic' in Indian music the tonic 'sa' is played throughout by the tanpura, and one never deviates from it, while in Western music the music can start in one key (with a particular tonic – such as C) and then 'modulate' to a different key, with a different tonic (such as G major, whose tonic is the fifth, or 'dominant' , of C major); the only requirement is that the tonic must return at the end of the piece. There are rules, though not very rigid ones, about how the tonic is to be approached, but unlike in Indian music these rules relate to harmonies and chord changes rather than individual notes.

A 'fugue' is a step beyond the canon, with somewhat less rigid rules and greater freedom. A fugue begins with a single voice, introducing a theme. Then as the voice continues, a second voice introduces the same theme in a different key, forming a counterpoint with the first voice. Similarly a third voice, a fourth, and so on can enter, again in various keys. After all the voices have been introduced, the form of the fugue is relatively unconstrained: each voice plays out its own melody in counterpoint with the rest. In the course of this, the theme can reappear in various forms: as it is, speeded up, slowed down, in 'inverted' form, and in any number of other ways. A listener to a fugue can have fun spotting new 'disguised' appearances of the theme. The fugue was also one of Bach's favourite forms of writing, an instance of his love of order and pattern. His two volumes entitled *'The Well-Tempered Clavier'* (of which more will be said shortly) contain 48 fugues, no two of them alike, and his last, unfinished work was called 'The Art of the Fugue' which was a series of short fugal pieces based on a rather simple theme and its inversion.

There are other musical patterns in Bach. For instance, *The Brandenburg Concertos* are probably the most widely heard of Bach's pieces, and they abound in such musical patterns. Many movements have fuguelike structures, for instance the third movement of the fourth concerto. The final movement of the third concerto is a masterpiece in getting maximum mileage out of minimum material: the fast, lively theme is extremely simple, and repeats again and again from different parts of the orchestra, interweaving with itself to form an effect quite mesmerising. Much of the same effect is seen in the finale of the first violin concerto, where again a cascade of triplets which by themselves form a reasonably uncomplicated melody build up in complexity through counterpoint with the orchestra.

The canons in particular are intriguing: Bach wrote them out as puzzle canons, with only the first part explicitly present and cryptic instructions to complete them with the second part.

'The Musical Offering' (which was dedicated to King Frederick the Great, and based on a theme composed by him) contains two fugues, ten canons and a trio sonata all incorporating the 'royal theme' in some form. The canons in particular are intriguing: Bach wrote them out as puzzle canons, with only the first part explicitly present and cryptic instructions to complete them with the second part. For the story of this particular opus, see the book by Hofstadter in the references.

A final example: the last fugue in 'The Art of the Fugue' is incomplete; it stops abruptly shortly after Bach introduces a new theme, consisting in modern notation of the notes B flat, A, C, B. To understand the point of this, one should know that in German notation B flat was called B, and B was called 'H'. Bach died shortly after encoding his name in his fugue, and without completing it.

The Equally Tempered Scale

We mentioned Bach's '*The Well-Tempered Clavier*' earlier; this is a set of two books, each containing 24 preludes and accompanying fugues for the keyboard (typically, in Bach's time, the harpsichord, though today they are often played on the piano). The work has immense musical significance, but Bach wrote this work to make a technical, as well as a musical point: the work was meant to show the advantages of a system of tuning the keyboard called equal temperament' (and this is what the title refers to – 'clavier' being a generic name for a keyboard instrument). Which brings us back to the question of what constitutes a musical scale.

A scale, in the broadest sense, is a collection of notes which can form the basis for a melody. The Pythagorean scale discussed above is a very good approximation to the scale that dominates world music, but it contains room for fine adjustments which are nevertheless perceptible to the trained ear. An example was adjusting the third note slightly to fall in consonance with the first. More generally, the pitch of a note as sung by a singer often depends on the notes that precede or follow it: in the west, for instance, musicians like to slightly sharpen (make higher) the 'leading note' or the note just below the tonic (B in the case of C major), and in Indian music the exact pitch of a note such as 'shuddh ga' depends on the raga that's being played, and depends on whether the singer is ascending or descending, and even on the particular performer. All these possible 'ga's' are sufficiently close to identify them as 'ga' but depending on the mood their pitch can vary minutely, so a mathematical theory of the ideal 'ga' becomes impossible in purely melodic (monophonic) music.

In harmonic or polyphonic music, however, we have notes being sounded simultaneously, and then comes the requirement that they should be consonant when sounded together. Much Western music is built upon chords and chord progressions, which are sequences of notes in small integer ratios played together or successively. For instance, the simplest chord would contain three notes (two notes, by definition, don't constitute a chord) and if we are to avoid the trivial octave (i.e. ratio of 1:2), the simplest ratios for these three are 4:5:6. Three notes in these ratios constitute a so-called 'major' chord: for instance, sa-ga-pa or C-E-G (which is called the C major chord).

When the notes are played successively, it sounds very similar to the Pythagorean scale, but when they are played together as chords, this scale sounds altogether smoother and richer. The whole Western concept of a melody eventually evolved around the chords of a scale.

The major chord by itself constitutes the basis of many well-known melodies. To take three examples: the opening of Bach's second violin concerto; the triumphant opening notes of the last movement of Beethoven's fifth symphony; the refrain of the Beatles' 'Ob-la-di, ob-la-da' , are all major triads C-E-G. But these three notes are not enough to constitute a scale. However, let us consider the two arguably most important notes in the scale, G (called the dominant, because it's a fifth above C, i.e. 3/2 times C) and F (called the sub-dominant, because it's a fifth below the next C, i.e. 2/3 times C′). If we demand that the major chords constructed on these notes, G-B-D and F-A-C, are also perfect 4:5:6 triads, then we have enough information to specify the frequencies of all the notes in the scale, and these notes are furthermore quite adequate for a perfectly respectable melody. To write out the frequencies of these notes explicitly, then, the 'harmonic' major scale is

$$1, 9/8, 5/4, 4/3, 3/2, 5/3, 15/8, 2$$

When the notes are played successively, it sounds very similar to the Pythagorean scale, but when they are played together as chords, this scale sounds altogether smoother and richer. The whole Western concept of a melody eventually evolved around the chords of a scale (C major, G major, F major), so melodies began to be constructed by skipping notes in the manner of a chord progression, rather than linearly ascending or descending a scale, and basing the successive groups of notes on different chords. A typical sequence of chord changes may go C-G-C-F-C-G-C: note that the C major chord dominates the melody and is expected to begin and end it, but in between the notes can organize themselves about other chords.

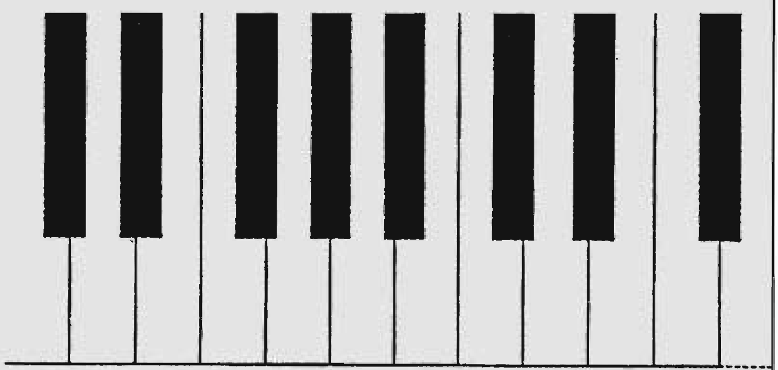
A lot of simple melodies can be written based purely on these principles (for instance, many Christmas carols, the Beatles tune mentioned above, and other popular tunes), and if that were all there was to the story there would be no problem. But we mentioned earlier the idea of modulating to different keys: when playing in C major, for instance, it is natural to modulate to the keys of G major or F major because their tonic chords are respectively the dominant and sub-dominant of C. So we have to consider how to construct a scale starting from G (say, G, A, B, C, D, E, F, G) rather than C#. One can do this in an exactly analogous manner (one can directly lift the ratios of various notes from the key of C and multiply them into the pitch of G, which is 3/2 above). One then runs into two problems:

1. The pitch of F works out very different from above: one gets 45/32, which is something like halfway between F and G above. So we call it by a new name, F (F-sharp). On a conventional piano keyboard, the notes C, D, E etc are white keys, and F# is a black key between the white keys F and G.

2. The other problem cannot be wished away so easily: the pitch of A works out almost the same, but slightly different. Rather than 5/3 as above it works out to 27/16. This means that if the keyboard is tuned to C major, and then played in G major, one note will be slightly wrong. The difference is too small to consider it a different note (as in F# above) but large enough to be noticeable to sensitive ears. The problem builds up, with a wrong note added with each modulation by a fifth: thus, D major has 2 wrong notes, and A major (A, B, C#, D, E, F#, G#, A) has 3 wrong notes (A, B, E). A piano tuned perfectly to C major would sound noticeably off-key played in A major.

Worse is to come: as we go up the fifths ladder and accumulate more and more 'sharp' notes, we reach a note called E# (in the key of F# major) which is very close to but a little lower than the natural F. Likewise if one goes down the fifths scale through the keys of F, B♭ flat (B flat – again, a 'black key' on the piano between A and B), etc, one reaches a note called G flat which is almost, but not quite, the same as F sharp.

The other scale that has taken root in the west, apart from the major scale, is called (naturally) the minor scale; the minor chord is basically a sort of inversion of the major chord, with the ratio of frequencies of the first two notes being 5:6 and of the second and third being 4:5. Just as we constructed the major scale about the major chord, we can construct the harmonic minor scale from the minor chord by demanding that the chords built on the tonic, subdominant and dominant all be minor chords. The scale of C major as calculated above serves fine as the scale of A minor when one starts from A rather than C, with the small flaw that the D is slightly off: if one changes that to 10/9 rather than 9/8, and then starts the scale from A, one gets the minor scale. The minor scale, too, is often used as the basis of a melody (for instance, the popular song 'Scarborough Fair'). The key of A minor is known as the relative minor to C major, since it uses the same notes, and C minor (=C, D, E♭, F, G, A♭, B♭, C) is known as the parallel minor to C major. Tunes in minor keys tend to sound a little



Layout of a musical keyboard

Tunes in minor keys tend to sound a little dark, while major keys sound more cheerful; this possibility is often exploited by composers.

dark, while major keys sound more cheerful. It is not invariably the case, but the possibility is often exploited by composers: for instance the triumphant sound of the opening of the last movement of Beethoven's 5th symphony is mainly because of the sudden change of keys from C minor to C major.

It seems, then, that a piano tuned to C major can only be played in C major, or at best in neighbouring keys like A minor, F major or G major; 'remote' keys will involve a profusion of black keys and 'wrong notes' on the keyboard. The problem does not arise with instruments like the violin, or with the human voice, since these can adjust pitches continuously and change according to the needs of the key being played. Thus composers of keyboard music, until the 1700s, were constrained by the limitations of tuning the keyboard and could not modulate their music to 'remote' keys.

This situation is often expressed by saying that 'the circle of fifths does not close'. By this is meant the fact that if one follows the procedure of raising a note by a fifth repeatedly, lowering it an octave when necessary to keep within the initial octave, one never recovers the original note. However, after twelve steps one does recover what is extremely close to the original note. More precisely,

$$(3/2)^{12} = 129.74 \ldots 2^7 = 128$$

so raising a note twelve fifths and lowering it seven octaves brings it to 1.013 ... times its original value. It would be nice if it were exactly the original value but that is a mathematical impossibility.

The suggestion that arose in the seventeenth century was to slightly flatten the fifth, to close the circle of fifths. In other words, define the C-G interval to be $2^{7/12} \sim 1.498$, rather than 1.5 exactly, and calculate all other notes by the same technique of moving up a fifth from the preceding note. It turns out that with this procedure one needs only twelve notes in the scale: B# is exactly the same as C, C# is exactly the same as D♭, and so on; and the ratio of any two successive notes is precisely $2^{1/12}$. Suppose one restricts oneself to intervals less than an octave (that is, frequency ratios between 1 and 2), and defines their combination modulo the octave: for instance, two fifths (3/2 and 3/2) combine to give the second (9/8), rather than 9/4. Then the set of all possible intervals forms a group, but this group has an infinite number of elements. The set of intervals that actually exist on a keyboard tuned to the natural scale cannot form a group, since most of the intervals when combined don't form other intervals which exist on the keyboard. The 'equally tempered' scale, however, does form a group with only twelve elements, $2^{n/12}$ where $n = 0, 1, 2, \ldots, 11$.

Of course, this plays havoc with all the nice integer ratios which we listed before: none of them survive, except the octave – indeed, they all become irrational. To suggest destroying the purity of the fifth and the fourth requires great boldness, and purists weren't pleased. The point of the suggestion was that the effect was actually not so bad as that of playing a scale on a piano tuned perfectly to a very remote scale. To most ears, the difference between these equally spaced 12 notes and the ideal ones is hardly perceptible. And while no key will now be perfectly in tune, no key will be more out of tune than any other. The question was whether it was worthwhile to sacrifice a small amount of purity of pitch in favour of vastly greater freedom in modulation.

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Bach was not the originator of this idea of equal temperament, but he was the first major composer to promote it vigorously. Before him, composers were content to compose keyboard music restricting themselves to keys close to C major. To emphasize the freedom offered by the new system, Bach wrote book I of *The Well–Tempered Clavier* : the book contained 24 preludes and 24 accompanying fugues, one in each of the major and minor keys. (Since there are now exactly twelve notes in the octave, there are only twelve distinct major keys and twelve minor keys: for instance F sharp major is no longer different from G flat major). Many years later he wrote another similar volume, which again

Suggested Reading

[1] D R Hofstadter, *Gödel, Escher, Bach: An Eternal Golden Braid* (Penguin, London, 1980): mainly a discussion of Gödel's theorem and artificial intelligence but an excellent place to start for appreciating Bach as well.

[2] Sir James Jeans, *Science and Music* (reprinted by Dover Publications, 1968): a classic on the subject.

traversed the keys in 24 preludes and fugues. This volume was untitled and is generally referred to as Book II of the WTC. *The Well-Tempered Clavier's* mission of promoting the equitempered scale succeeded: it is the only system used to tune keyboards today. But musically, too, the work profoundly influenced a number of composers. Beethoven, for instance, was made by his teacher to memorize the whole work, and for the rest of his life kept a portrait of Bach on his desk; his late works display a preoccupation with fugal writing. Mozart discovered Bach late in life but the discovery sparked an interest in contrapuntal writing evident in his later works. Chopin, in a reference to the WTC, wrote a set of 24 preludes, with a similar tour of the keys. He told students to 'always play Bach'. In this century, Shostakovich wrote a set of 24 preludes, and later another set of 24 preludes and fugues: his version of the WTC, but little-known, and recorded for the first time only in the 1980s.

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And on this note we end this brief survey of Bach: little known in his time, regarded as old-fashioned and backward-looking by his sons and others, but one of the most influential composers ever in more ways than one and, to many admirers, the epitome of perfection in music.