Systems and Control Engineering

2. Basic Concepts of Systems

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In this part we define a number of properties of systems and describe their mathematical representation.

Properties of Systems

Consider *Figure* 1. The input and the output are essential in characterising the system, as we have seen in the automobile example given in Part 1. In other words, a system may be thought of as a mapping from the set of input variables to the set of output variables. For the time being, we shall consider the systems which have a single input and a single output (SISO systems). We shall take the more general case of multiple-input-multiple-output (MIMO) systems in due course. In what follows, we use the following symbols.

- $u(t)$: input signal
- $y(t)$: output signal
- $r(t)$: reference signal

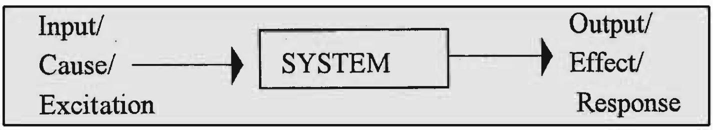
Occasionally, we drop the argument t for the sake of brevity.

Let us first look at some of the basic definitions and characteristics of systems.

Systems With and Without Memory:

A system is called *memoryless* if its output $y(t_1)$ at time $t = t_1$ depends only on the input applied at that instant; $y(t_1)$ is independent of the input applied before $t = t_1$. Otherwise, the system is said to have memory. A memoryless system

Figure 1.



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can be modelled by an algebraic equation. A system with memory, also known as *dynamical system*, is usually described by a differential equation.

Example 1. An electrical circuit with only resistances is memoryless, whereas one with resistance and capacitance or inductance has memory.

Example 2. A simple logic gate such as a NOT/OR/AND is memoryless whereas a flip-flop is a dynamical system.

Causal and Non-causal Systems

A system is said to be *causal* if its output $y(t_1)$ at $t = t_1$ depends only on the input applied until that instant, but not later. In other words, a causal system does not anticipate inputs. On the contrary, if output of the system $y(t_1)$ depends on an input that is to be applied at a later time $t_2 > t_1$, then the system is said to be *non-causal* or *anticipatory*.

A system with memory, also known as dynamical system, is usually described by a differential equation.

No physical system can be anticipatory.

Linear and Nonlinear Systems

A system is said to be *linear* if and only if it satisfies the following properties.

[1] If an input u_i gives an output y_i, then an input that is k times u_i, gives an output which is also k times y_i. This property is called *homogeneity*.

[2] If each of n inputs $u_i, i = 1, 2, \ldots, n$ causes n different outputs y_i respectively, then an input that is the sum of these n inputs, i.e., $u = u_1 + u_2 + \ldots + u_n$, causes an output y which is the sum of the n outputs, i.e., $y = y_1 + y_2 + \cdots + y_n$. This property is called *additivity*.

These two properties may be combined as follows to obtain the total response of the system due to n different excitations. An input

$$u = k_1 u_1 + k_2 u_2 + \ldots + k_n u_n,$$

causes an output

$$y = k_1 y_1 + k_2 y_2 + \ldots + k_n y_n.$$

This is known as the principle of *superposition*. Thus a system is said to be linear if and only if the principle of superposition is applicable. Otherwise, it is nonlinear.

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Time-Invariant and Time-varying Systems

A time-invariant system is one whose characteristics do not change with time.

Even though most systems are time-varying in the long run, we assume that they are time-invariant relative to the short time-constants of the systems. (For a definition of time-constant please refer to the next section).

Mathematical Representation of Systems

In the present article, we restrict ourselves to linear time-invariant (LTI) systems. Let us assume that an unit impulse $\delta(t)$ (also called the Dirac's delta function) is fed as input to a system. The corresponding output of the system is called the impulse response, denoted by $h(t)$.

Any arbitrary signal (or function) $u(t)$, for example as shown in *Figure* 2, may be expressed as

$$u(t) = \Sigma_{j=0}^{\infty} \delta(t - j\epsilon) u(j\epsilon)\epsilon \; \forall t \geq 0 \qquad (1)$$

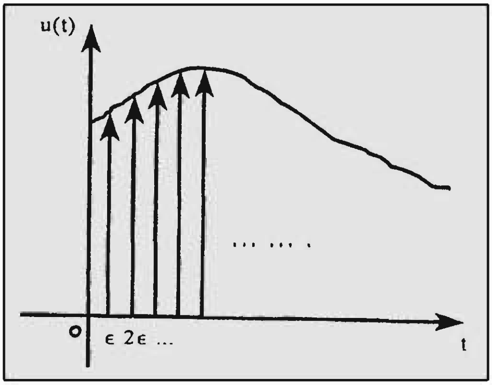

Figure 2.

where ϵ is the infinitesimal time interval between two successive impulses.

This expression is analogous to the trapezoidal rule for numerical integration.

If we define $\lambda = j\epsilon$ so that $\epsilon = d\lambda$, (1) may be rewritten as

$$u(t) = \int_{\lambda=0}^{\infty} \delta(t-\lambda)u(\lambda)d\lambda \tag{2}$$

in the limit $\epsilon \to 0$.

Let us now use the properties of linearity and time-invariance and derive an expression for the output of the system subjected to any arbitrary input expressed by (2).

Time-invariance: If $\delta(t) \to h(t)$ (impulse response), then $\delta(t-\lambda) \to h(t-\lambda)$

Linearity: $u(t) = \int_{\lambda=0}^{\infty} \delta(t-\lambda)u(\lambda)d\lambda \to y(t)$

$= \int_{\lambda=0}^{\infty} h(t-\lambda)u(\lambda)d\lambda.$

Further, if we assume that the system is causal, $u(t-\lambda) = 0 \forall \lambda > t$. Hence,

$$y(t) = \int_{\lambda=0}^{t} h(t-\lambda)u(\lambda)d\lambda. \tag{3}$$

This integral description, called the convolution integral, is the simplest and intuitively appealing description considering the properties of linearity, time-invariance and causality. However, this is not a general description as it fails to accommodate initial conditions, if any.

Example 1: Consider a simple electrical system shown in *Figure* 3. (This is also called a physical model in systems parlance since it may as well be an analog of, say, a thermal system.)

If an unit impulse voltage $\delta(t)$ is applied to the circuit, at $t = 0^+$ the capacitor gets charged to 1 volt and then discharges through the resistance exponentially. Thus, the impulse response is $h(t) = e^{-t}$.

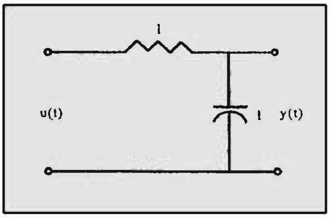

Figure 3.

Therefore $y(t)$, for any input $u(t)$ is

$$y(t) = \int_0^t e^{-(t-\lambda)} u(\lambda) d\lambda \tag{4}$$

This is the convolution integral description of the above electrical circuit. However, if we differentiate both sides of (4) with respect to time,

$$\dot{y}(t) = \int_0^t \frac{d}{dt} e^{-(t-\lambda)} u(\lambda) d\lambda + e^{-(t-\lambda)} u(\lambda)|_{\lambda=t}$$

or,

$$\dot{y}(t) + y(t) = u(t) \tag{5}$$

A differential equation description, which can be obtained from a convolution integral, is more general in that non-zero initial conditions can be accounted for without any modification. Hence, we describe any system by means of its differential equation, hereafter. We mention here that a system is said to be *lumped* if it can be described by ordinary differential equations and *distributed* if the description is given by partial differential equations.

We restrict ourselves to lumped systems in this article.

The solution to the differential equation (5) may be easily obtained as

$$y(t) = y(0)e^{-t} + \int_0^t e^{-(t-\lambda)} u(\lambda) d\lambda \tag{6}$$

Notice the convolution integral appearing as the second term on the RHS of (6). Also a term due to the initial condition appears. Thus, the complete response has two parts – the natural response, due to the initial condition and the forced response due to an externally applied signal. More generally, if the circuit has a resistance R and a capacitance C, then

$$y(t) = y(0)e^{-\frac{t}{RC}} + \int_0^t e^{-\frac{(t-\lambda)}{RC}} u(\lambda) d\lambda \tag{7}$$

A system is said to be *lumped* if it can be described by ordinary differential equations and *distributed* if the description is given by partial differential equations.

The product RC has the units of time and is denoted by τ. It is an index of the exponential change in the response $y(t)$, particularly the decay of $y(0)$. In an interval $t = \tau$, the natural response decays approximately by 36.8 %, and this interval is defined as the *time constant* of the system.

The natural response is also called the transient response and the forced response is called the steady-state response. It is easy to see that the natural response decays to almost zero in an interval $t = 6\tau$, justifying the name transient. Another point one should note is that such an interval of about 6τ is an *infinite* time in engineering practice! We leave it to the reader to guess what would be a typical time constant.

As long as the equations are linear, we can apply the Laplace transformation techniques, to obtain the solution without much difficulty.

A system is said to be of order *n* if the order of the highest derivative present in the differential equation is *n*.

A system is said to be of order n if the order of the highest derivative present in the differential equation is n. An n^{th} order system has n initial conditions. The Laplace transform converts the differential equation in t to an algebraic equation in the complex variable s.

If we take the Laplace transform of the solution in (6), we get

$$Y(s) = y(0)H(s) + H(s)U(s) \tag{8}$$

where $H(s) = \mathcal{L}\{e^{(-t)}\}$

The ratio of $Y(s)$ to $U(s)$ with all initial conditions set to zero is defined as the *transfer ratio* or *transfer function.* Thus, from (8), the Laplace transform of the impulse response is the transfer function. Notice that the transfer function is always a rational function of two polynomials in s. Let

$$H(s) = \frac{N(s)}{D(s)}$$

The roots of the numerator polynomial $N(s)$ are called *zeros* since $H(s) = 0$ if s takes any of these values. The roots of $D(s)$ are called *poles* since $H(s) = \infty$ and the function 'blows up' in the three dimensional representation of the magnitude of the transfer function as a function of complex variable s. Poles and zeros are called critical frequencies. At other complex frequencies, the transfer function has a finite, nonzero value.

Poles and zeros are called critical frequencies.

In the previous example,

$$\begin{aligned} H(s) &= \mathcal{L}\{(e^{-t})\} \\ &= \frac{1}{s+1}. \end{aligned}$$

i.e., the transfer function has no zeros but has a pole at $s = -1$.

In the absence of any physical input, since $y(0)$ is a constant, say y_0, the transient response is determined solely by the poles of the transfer function. Accordingly, the denominator polynomial $D(s)$ is called the *characteristic polynomial* and the equation $D(s) = 0$ is called the *characteristic equation.*

Example 2: Let a system be described by the differential equation

$$\ddot{y} + 3\dot{y} + 2y = 3u \tag{9}$$

where the initial conditions are $y(0)$ and $\dot{y}(0)$.

Taking Laplace transforms and rearranging, we get two terms : the transient response, $Y_{\text{tr}}(s)$, and the steady-state response, $Y_{\text{ss}}(s)$.

$$Y(s) = \frac{(s+3)y(0) + \dot{y}(0)}{s^2 + 3s + 2} + \frac{3U(s)}{s^2 + 3s + 2} \tag{10}$$

The transfer function has no zeros, but has two poles at $s = -1$ and $s = -2$.

The first term in (10), denoted by $Y_{\text{tr}}(s)$, gives the transient response of the form $Ae^{-t} + Be^{-2t}$ where A and B are the residues obtained by expanding $Y_{\text{tr}}(s)$ into partial fractions.

If the set of poles is a subset of the set of modes, then the system is said to be completely characterized.

If we assume that the input $u(t)$ is a unit step input, i.e., $U(s) = \frac{1}{s}$, the forced response takes the form $C + De^{-t} + Ee^{-2t}$.

The complete response is the sum of forced response and transient response i.e.,

$$y(t) = P + Qe^{-t} + Re^{-2t}$$

The zeros and poles also play a role in the steady-state response. However, since $H(s)$ is multiplied by another function $U(s)$, the set of poles and the set of zeros are modified. Since we assumed $U(s) = \frac{1}{s}$, the set of poles is $\{0, -1, -2\}$. This modified set is called the set of *modes*. Notice that, then, there is a possibility that a pole of the transfer function may be cancelled by a zero of $U(s)$. In such a case not all the roots of the characteristic equation determine the forced response. Accordingly, we say that the system is not *completely characterized.* In other words, if the set of poles is a subset of the set of modes, then the system is said to be completely characterized.

With this background, we may identify three types of problems

1 Given $u(t)$, and the system description (by means of either a transfer function or a differential equation), find $y(t)$. This is the *analysis* problem.

2 Given $u(t)$ and $y(t)$ (perhaps including the initial condition), find a suitable system (i.e., its transfer function or differential equation which meets the specifications). This is the *synthesis* problem.

3 Given the system and a desired response $y(t)$, find $u(t)$ such that the system meets the specified requirement. This is the control system *design* problem. This is elaborated in the next part of the series.

Suggested Reading

[1] B C Kuo, *Automatic Control Systems*, 7/e, Prentice Hall of India, 1997.

[2] W Brogan, *Modern Control Theory*, Prentice Hall, Englewood Cliffs, NJ, 1991.

[3] D'Azzo and Houpis, *Linear Control System Analysis and Design*, 4/e, McGraw Hill International Edition, 1995.

[4] F M Callier, C A Desoer, *Linear System Theory*, Narosa Publishing House, 1992. (This is for the advanced reader).

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