Whoever said Nothing is Impossible?

Three Problems from Antiquity

Shashidhar Jagadeeshan

1. Prologue

In the history of mathematics there have been periodical discoveries which have changed the very way we think about mathematics. For example, the discovery of non-Euclidean geometries in the early nineteenth century made us question the link between mathematics and reality. Gödel's work, in this century, not only introduced the notion of undecidability, but also raised very serious questions about the axiomatic foundations of mathematics. In a similar vein the discovery in the nineteenth century that certain problems in mathematics are impossible to solve, revealed a totally new aspect to mathematics. Yes! that's right, there are problems in mathematics that are impossible to solve. It's not a matter of being clever (you might be a Gauss or a Ramanujan) or how hard you work, but given certain constraints these problems are impossible to solve. What's more, we can prove that this is so! The present article will focus on proving the following three problems of antiquity:

1. *The duplication of a cube*, or the problem of constructing a cube having twice the volume of a given cube.
2. *The trisection of an angle*, or the problem of dividing a given arbitrary angle into three parts.
3. *Squaring a circle*, or the problem of constructing a square whose area is equal to that of a given circle.

are **impossible** to solve; under the constraint that we use only a straight edge and compass. The epilogue will discuss some other impossibilities in mathematics. Before I define the problems more clearly and give an outline of their proofs, I would like to give a brief historical introduction to them.

All three problems discussed here have their origins in Greek geometry around the 5th century BC. These problems have

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Box 1.

The following is the translation (taken from, [5]) of an ancient document supposedly written by Eratosthenes to King Ptolemy III about the year 240 B.C. :

To King Ptolemy, Eratosthenes sends greetings. It is said that one of the ancient tragic poets represented Minos as preparing a tomb for Glaucos and as declaring, when he learnt that it was hundred feet each way: "Small indeed is the tomb thou hast chosen for a royal burial. Let it be double [in volume]. And thou shalt not miss that fair form if thou quickly doublest each side of the tomb." But he was wrong. For when the sides are doubled, the surface [area] becomes four times as great and the volume eight times. It became a subject of enquiry among geometers in what manner one might double the given volume without changing shape. And this problem was called the duplication of the cube, for given a cube they sought to double it ... .

become famous because they are impossible to solve within the constraints imposed. The resolution of these problems had to wait for nearly 2200 years, for the invention of abstract algebra. In the meantime many great mathematicians, amateurs and mathematical cranks have had a go at them without much success. To this day mathematics departments around the world receive letters from people (unfamiliar with the fact that these problems are impossible to solve), claiming to have solved them! Like many other long-standing unsolved problems, these too have inspired the discovery and creation of a lot of mathematics such as that of the conic sections, many cubic and quartic curves and later on algebraic numbers and group theory.

Problem 1 might well have originated in the words of an obscure ancient Greek poet representing the mythical King Minos as being dissatisfied with the size of the tomb erected to his son Glaucos (see *Box* 1). Minos had ordered the tomb be doubled in size. The poet proceeds to say that Minos claimed this could be accomplished by doubling the length of each side. Obviously neither Minos nor the poet passed high school algebra! Attempts to correct the mathematics of the poet led geometers to study the problem of doubling the cube. There is another story related to this problem, where it is said, that the Delians were instructed by their oracle to double the size of Apollo's cubical altar, so as to get rid of a certain pestilence. This is why this problem is also referred to as the *Delian Problem.*

The origins of Problem 2 are most likely in the Greeks' attempt to construct regular polygons. In the construction of a regular nine-sided polygon, one needs to trisect an angle of sixty degrees. In fact this problem has attracted a large number of so called 'angle trisectors', who to this day attempt to trisect an angle.

The history of Problem 3 is tied up with that of finding the area of a circle, and information about this is found in the Rhind (or Ahmes) Papyrus (one of the best known mathematical manuscripts, purchased in 1858 in Egypt by A Henry Rhind and later acquired by the British Museum). Here it states that the area of a circle is equal to that of a square whose side is the diameter diminished by one ninth. This is of course based on the crude approximation of π to be 3.1604. .. Since these problems fascinated many mathematicians and amateurs, the problem solvers themselves acquired names! For example, in Greek times 'circle squarers' were referred to by the term 'tetragonidzein', which means to occupy oneself with the quadrature.

As you can see these problems come with a very rich history, and have many fascinating anecdotes connected with them, so interested readers can pursue them in standard books on the history of mathematics [1]. The problems were finally settled in the nineteenth century. P Wantzel showed in 1837 that Problems 1 and 2 are impossible to solve and in 1882, Problem 3 was shown to be impossible to solve by F Lindemann.

2. Ruler and Compass Constructions

Anyone who has done high school geometry is familiar with constructions using only a ruler and compass. You may well remember constructions like bisecting an angle, drawing the perpendicular bisector of a given line segment and so on. Apart from some of these easy and obvious constructions it's amazing what one can do with these tools. For example, one can construct a square having the same area as that of a lune, i.e., a plane figure bounded by two circular arcs! For an excellent account and proof of this fact, I would urge you to read [2]. It is not clear why the Greeks imposed these constraints on themselves. Perhaps it was due to the notion of Plato that the only 'perfect' figures were the straight line and the circle, or it was an intellectual game, with very precise rules. It must be noted here that the Greeks themselves did not always restrict themselves to these tools and did not hesitate to use other instruments. We will discuss this briefly in the epilogue.

So what then are the precise rules involved in constructions using a ruler and a compass? Remember by ruler we mean a 'straight edge'; one cannot use it to make measurements or transfer lengths, only draw straight lines. If you notice all constructions using ruler and compass start with a given set $\mathcal{S} = \{q_1, q_2, \ldots, q_n\}$ of points and we use the ruler and compass in the following two ways:

> **Operation 1**(ruler) Through any two points in $\mathcal{S}$ draw a straight line.
>
> **Operation 2** (compass) Draw a circle, whose centre is a point of $\mathcal{S}$, and whose radius is equal to the distance between some pair of points in $\mathcal{S}$.

We note here that the Greeks preferred a more restricted compass than the one we allow. Their compass was a 'collapsing' one which when lifted from the paper collapsed, not allowing one to maintain a fixed radius. Their compass allowed one to place the compass point at one point of $\mathcal{S}$ and the pencil tip at another point of $\mathcal{S}$ and draw a circle or an arc. It can be shown that every construction that can be performed by a noncollapsing compass such as ours can be performed by a collapsing one (see Exercise 5.11, pp.59–60 of [3]).

While doing ruler and compass constructions you might have realized that all constructions involve the intersections of a line with another line, a line with a circle or a circle with another circle. How do we formalize these ideas? We start by making two definitions.

Definition 2.1. The points of intersection of any two distinct lines or circles using operations 1 or 2, are said to be *constructible in one step* from $\mathcal{S}$. A point p is said to be *constructible* from $\mathcal{S}$ if there is a finite sequence $p_1, p_2, \ldots, p_n = p$ of points such that for each $i = 1, \ldots, n$ the point p_i is constructible in one step from the set $\mathcal{S} \cup \{p_1, p_2, \ldots, p_{i-1}\}$.

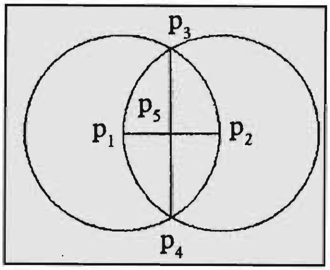


Figure 1. Bisecting a line segment using ruler and compass.

Definition 2.2. Let γ be a real number with absolute value $|\gamma|$. Then γ is said to be *constructible* if we have two constructible points p_i,p_j whose distance apart is $|\gamma|$ units, the starting set $\mathcal{S}$ for the points p_i, p_j being the set of points $\{p_0, p_1\}$ whose distance apart is 1 unit.

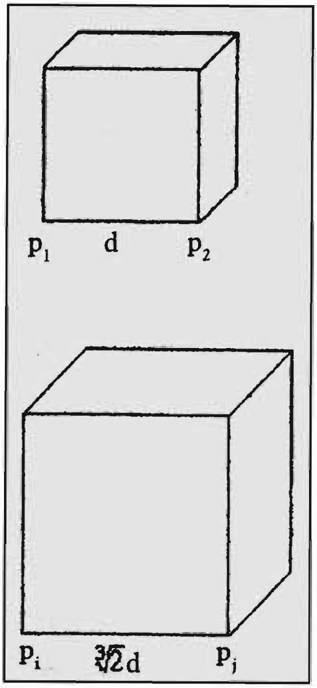
Let us now use a simple construction as an example to illustrate our formal set-up. Suppose we are given two points p_1, p_2. Let $\mathcal{S} = \{p_1, p_2\}$. (*Figure* 1)

1. Draw the line p_1p_2 (operation 1);
2. Draw the circle centre p_1 of radius p_1p_2 (operation 2);
3. Draw the circle centre p_2 of radius p_1p_2 (operation 2);
4. Let p_3 and p_4 be the points of intersection of these circles;
5. Draw the line p_3p_4.(operation 1);
6. Let p_5 be the intersection of the lines p_1p_2 and p_3p_4.

Then the sequence of points p_3, p_4, p_5 defines a construction of the midpoint of the line p_1p_2.

We are now ready to see what all this formal stuff looks like when we consider the three problems of antiquity.

Figure 2. Doubling the cube.



Doubling the Cube. We are given a cube and asked to construct a cube with double its volume. Given two points p_1 and p_2 whose distance apart is equal to one side of the original cube, our task is then to construct two points p_i and p_j whose distance apart is exactly $\sqrt[3]{2}$ times the distance between p_1 and p_2(*Figure* 2).

Trisecting an Angle. We are given points p_1, p_2 and p_3 which determine the angle and are asked to construct a point p_i using only operations 1 and 2, such that angle $p_1p_2p_i$ is exactly one third that of the angle $p_1p_2p_3$. (*Figure* 3)

Squaring the Circle. Given two points p_1 and p_2, whose distance apart is the radius of the circle, construct two points p_i and p_j whose distance apart is $\sqrt{\pi}$ times the distance

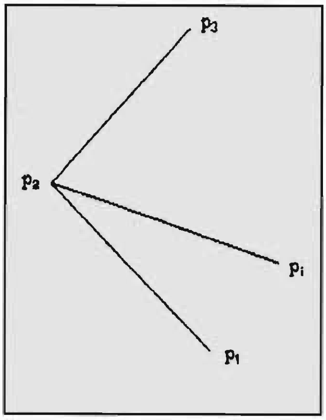


Figure 3. Trisecting an angle.

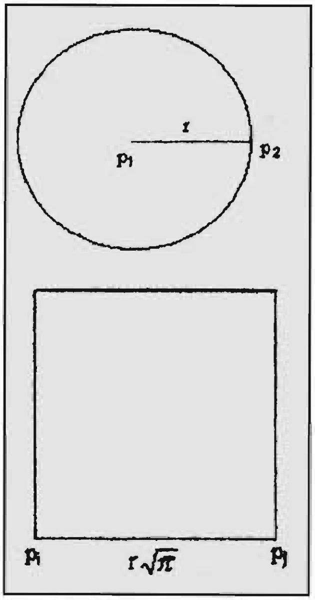
between p_1 and p_2.(*Figure* 4)

3. How Does Algebra Come into the Picture?

We will now show how we convert these problems into the language of algebra and can also sympathize with why the Greeks had so much trouble with them.

As mentioned earlier, not only was the word algebra 'Greek' to the Greeks, they also did not realise that these problems were impossible to solve. To convert these problems into algebra, the reader will now have to be patient as we build up the required background. For reasons of brevity we can't give a comprehensive background and would recommend any standard text in algebra say [4]. Moreover, we will not give complete proofs, but will only give a sketch. Many of the definitions and statements of theorems are taken directly from the books referred below. The algebra that is used is based on the work of many great algebraists, like Abel, Galois, Lagrange and more recently Artin. The material is also the beginnings of the beautiful area of mathematics known as Galois theory.

Figure 4. Squaring the circle.



Our aim is to characterize constructible numbers algebraically. In other words convert the geometric idea of a constructible number as defined in Definition 2.2 to that of an algebraic one and understand it in the language of algebra. Actually, looked at correctly algebra comes in immediately!

Theorem 3.1. *The set of constructible numbers is a subfield of R. Furthermore every rational number is constructible and if γ is a positive constructible number then so is $\sqrt{\gamma}$.*

Proof: The first part follows from the well known fact that if α and β are constructible, then so are $\alpha+\beta$, $\alpha\beta$ and α/β. If we are given a unit length, then we can construct all the rationals. Once a given length $\gamma > 0$ is constructed, then it is fairly easy to construct (using ruler and compass) $\sqrt{\gamma}$.

□

By repeated applications of the above theorem, for example, one can prove that the number

$$\sqrt[4]{13} + \frac{4}{3}\sqrt{\sqrt{6} + \sqrt{1 + 2\sqrt{7}}}$$

is constructible.

The theorem tells us that the class of constructible numbers is large. In fact our aim is to find the ones that are **not** constructible. More specifically, if we are able to double the cube then $\sqrt[3]{2}$ has to be constructible, and if we can square a circle $\sqrt{\pi}$ would be constructible. Before we proceed we need to understand the notion of a number being algebraic over a field. The set of constructible numbers is a subset of the set of algebraic numbers, so information about the latter will be useful to understand the former.

Definition 3.1. A number $\alpha \in \mathbb{C}$ is said to be *algebraic* over a field F contained in $\mathbb{C}$ if there exists a non-zero polynomial $f(x) \in F[X]$ such that α is a zero of $f(x)$. If no such polynomial exists then α is said to be *transcendental* over F.

In other words, if α is algebraic over F there exists a polynomial $f(X) = a_0 + a_1X + \;..+ a_nX^n$ whose coefficients $a_0, a_1, \ldots a_n$ are all in F with at least one of these coefficients being non-zero and $f(\alpha) = 0$. For example $\sqrt{2}$ is algebraic over $\mathbb{Q}$, since it satisfies the polynomial $X^2 - 2 = 0$.

Recall that a polynomial is monic if its leading coefficient is 1.

Definition 3.2. Let α be algebraic over $F \subseteq \mathbb{C}$. Then there exists a unique monic polynomial $f(X)$ over F of least degree such that $f(\alpha) = 0$ and it is called the *irreducible polynomial of α over F* , and is denoted irr (α, F).

For example, $X^2 - 2$ is the irreducible polynomial of $\sqrt{2}$ over $\mathbb{Q}$. We denote the degree of $\mathrm{irr}(\alpha, F)$ by $\deg(\alpha, F)$.

Definition 3.3. Let F be a field. A non constant polynomial $f(X) \in F[X]$ is said to be *irreducible over F* if it has no factors in $F[X]$.

The following theorem connects the notion of the irreducible polynomial of a number and that of an irreducible polynomial over a field. Moreover given $\alpha \in \mathbb{C}$ algebraic over F, it gives us a way of computing $\operatorname{irr}(\alpha, F)$.

Theorem 3.2. *Let F be a subfield of $\mathbb{C}$ and let $\alpha \in \mathbb{C}$ be algebraic over F. The following conditions on a polynomial $f(X) \in F[X]$ are equivalent:*
(i) $f(X) = \operatorname{irr}(\alpha, F)$.
(ii) $f(\alpha) = 0$ and $f(X)$ is monic irreducible over F

In general it is not easy to find out if a given polynomial is irreducible or not. One powerful criterion is Eisenstein's criterion (see *Box* 2 for details).

Every time we have a field F embedded in another field K, i.e.,$F \subseteq K$, we obtain a vector space K over F. The dimension of K over F is denoted by $[K : F]$. The field K is called the *extension field* of F. The next theorem tells us when a given $\alpha \in K$ is algebraic over F.

Box 2.Eisenstein's Irreducibility Criterion

Let

$$f(t) = a_0 + a_1 t + \ldots + a_n t^n$$

be a polynomial over $\mathbb{Z}$. Suppose that there is a prime q such that

1. q does not divide a_n
2. q divides $a_i (i = 0, \ldots, n-1)$
3. q^2 does not divide a_0

Then f is ireducible over $\mathbb{Q}$.

Examples of polynomials where Eisenstein's criterion can be used to check irreducibility are:

$X^3 - 2$ and $X^3 + 3X^2 - 3$. In the first case $q = 2$ and in the second $q = 3$.

Theorem 3.3. *Let F be a subfield of a field K with $[K : F] = n$ where n is a positive finite integer. Then every number $\alpha \in K$ is algebraic over F and $\deg(\alpha, F) \leq n$.*

One can extend a given field F by 'attaching' a single element α (not in F, but in K) to F.

Definition 3.4. *Let F be a subfield of $\mathbb{C}$ and let α be algebraic over F with $\deg(\alpha, F) = n$. The extension of F by α is the set $F(\alpha) \subseteq \mathbb{C}$ where $F(\alpha) = \{b_0 + b_1\alpha + \ldots + b_{n-1}\alpha^{n-1} : b_0, b_1, \ldots, b_{n-1} \in F\}$.*

Theorem 3.4. *Let F be a subfield of $\mathbb{C}$ and let $\alpha \in \mathbb{C}$ be algebraic over F with $\deg(\alpha, F) = n$. Then*
(i) $F(\alpha)$ is a finite dimensional vector space over F with basis $\{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}$. The dimension n of $F(\alpha)$ over F as a vector space is denoted by $[F(\alpha) : F]$.
(ii) $F(\alpha)$ is a field.

For example the set $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is a field extension of $\mathbb{Q}$ of degree 2. Actually for our purpose we will be mainly interested in extensions of fields of degree 2, the so called quadratic extensions.

Now whenever we perform a construction using ruler and compass it is not enough to have an extension of the rationals by a single element; in fact we have to iterate the process to obtain a sequence of extensions. For example to find a suitable extension of $\mathbb{Q}$ where

$$\sqrt[4]{13} + \frac{4}{3}\sqrt{\sqrt{6} + \sqrt{1 + 2\sqrt{7}}}$$

lives, we need the following sequence of field extensions :
$F_0 = \mathbb{Q}$, $F_1 = F_0(\sqrt{7}), F_2 = F_1(\sqrt{1 + 2\sqrt{7}}), F_3 = F_2(\sqrt{6})$,

$$F_4 = F_3(\sqrt{6 + \sqrt{1 + 2\sqrt{7}}}), F_5 = F_4(\sqrt{13}), F_6 = F_5(\sqrt{\sqrt{13}}).$$

We now discuss briefly what happens if we have a tower of distinct subfields of $\mathbb{C}$, $E \subseteq F \subseteq K$ We notice that there are three vector spaces that are involved here. F over E, K over F and K over E.

It is natural to ask if, say, the dimensions of the vector

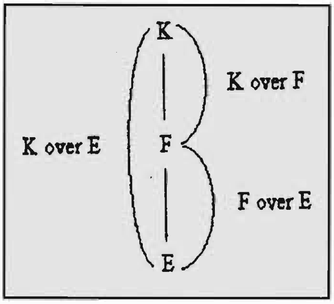


Figure 5. Tower of subfields of $\mathbb{C}$.

spaces F over E and K over F are finite, what is the dimension of K over E ? The following theorem gives the answer.

Theorem 3.5. *Consider a tower of subfields of $\mathbb{C}$, $E \subseteq F \subseteq K$ If the vector spaces F over E and K over F have finite dimensions, then so does K over E and $[K : E] = [K : F][F : E]$.(Figure 5)*

4. The Proofs of the Impossibilities at Last !

We are now almost ready to prove the impossibilities of the three constructions – we just need the following theorem that does the trick! The background material from the previous section was necessary and hopefully sufficient to understand this theorem. We have noticed from the example above that having a sequence of quadratic extensions produces a constructible number. The question then is : 'is there any other way to produce constructible numbers ?'. The answer is 'no'!

Theorem 4.1. *The following are equivalent :*

(i) The number γ is constructible.

(ii) There exists a sequence of positive real numbers $\gamma_1, \gamma_2, ..\ \gamma_n$ such that
$\gamma_1 \in F_1$ *where* $F_1 = \mathbb{Q}$,
$\gamma_2 \in F_2$ *where* $F_2 = F_1(\sqrt{\gamma_1})$,

$\gamma_n \in F_n$ *where* $F_n = F_{n-1}(\sqrt{\gamma_{n-1}})$,
and, finally,
$\gamma \in F_{n+1}$ *where* $F_{n+1} = F_n(\sqrt{\gamma_n})$.

Proof: (ii) implies (i) follows from Theorem 3.1 and induction on n. The proof that (i) implies (ii) rests on the fact that to produce a new constructible point one needs the intersection of either a line with a line, or a line with a circle or that of a circle with a circle. In each of the above cases at worst one may need extra square roots to describe the new point constructed, in other words a quadratic extension. We illustrate with an example below. □

Example 4.2. *Start with two points $P_0 = (0,0)$ and $P_1 = (1,0)$ at a distance 1 unit apart. Draw circles with centres at P_0 and P_1 with radius P_0P_1 then the points of intersection P_2 and P_3 can be computed to be $(1/2, \sqrt{3}/2)$ and $(1/2, -\sqrt{3}/2)$ respectively involving nothing more than $\sqrt{3}$. (Figure 6).*

Corollary 4.3. *If γ is constructible, then γ is algebraic and* $\deg(\gamma, \mathbb{Q}) = 2^s$ *for some integer $s \geq 0$.*

The proof follows from the above Theorems 4.1 and 3.5.

Corollary 4.4. *If a real number γ satisfies an irreducible polynomial over $\mathbb{Q}$ of degree n, and if n is not a power of 2, then γ is not constructible.*

This is an immediate consequence of the above corollary.

We are now really ready to prove that the three great problems of antiquity are impossible to solve!

Theorem 4.5. *The cube cannot be duplicated using ruler-and-compass constructions.*

Proof: As discussed earlier, if we could duplicate the cube then $\sqrt[3]{2}$ would be constructible. However $\sqrt[3]{2}$ satisfies the

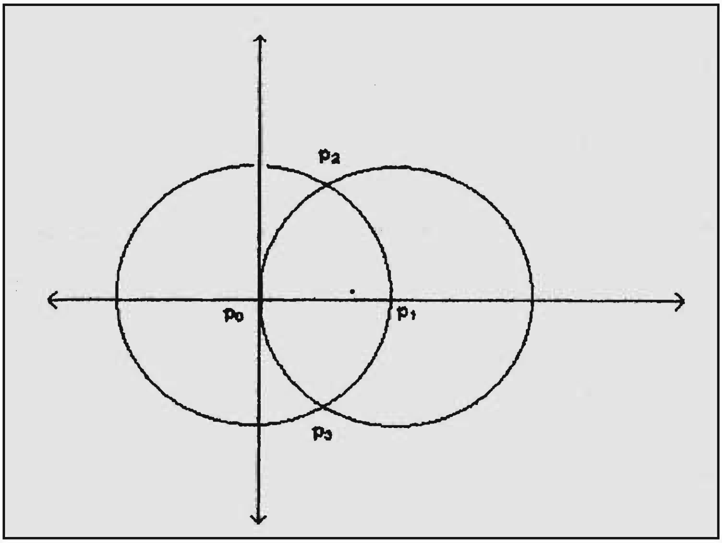


Figure 6.

monic polynomial $X^3 - 2 = 0$ over $\mathbb{Q}$, which by Eisenstein's criterion is irreducible over $\mathbb{Q}$. But the degree of $X^3 - 2 = 0$ is 3 and by Corollary 4.4, $\sqrt[3]{2}$ is not constructible. Hence it is impossible to duplicate the cube using ruler-and-compass constructions.

We now show that the angle $\pi/3$ cannot be trisected by a ruler-and- compass construction. Thus there is no ruler-and-compass construction which will provide a trisection of every angle. This leaves the possibility of being able to trisect 'most' angles, with the method failing only for a finite number of angles. However it turns out that once we have established the fact that $\pi/3$ is not constructible, it is quite easy to deduce that there are infinitely many angles that cannot be trisected (see [5], p.106).

Theorem 4.6. *The angle $\pi/3$ cannot be trisected using ruler-and-compass constructions.*

Proof: If the angle $\pi/3$ could be trisected, then the angle $\pi/9$ can be constructed. Using a right angled triangle it is easy to see that the length $\cos(\pi/9)$ and hence $\beta = 2\cos(\pi/9)$ can be constructed. Recalling the fact that $\cos(\pi/3) = 1/2$ and the trignometric identity $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$, we see that β satisfies the monic polynomial $\beta^3 - 3\beta - 1 = 0$. Now the polynomial $f(X) = X^3 - 3X - 1$ is irreducible over $\mathbb{Q}$, since the polynomial $f(X+1) = X^3 + 3X^2 - 3$ is irreducible over $\mathbb{Q}$, by Eisenstein's criterion. But once again the degree of $f(X) = X^3 - 3X - 1$ is 3 and by Corollary 4.4, β is not constructible. Hence $\pi/3$ cannot be trisected. $\square$

Theorem 4.7. *The circle cannot be squared using ruler-and-compass constructions.*

Proof: As discussed earlier if we could square the circle, then $\sqrt{\pi}$ would be constructible and hence by Corollary 4.3 algebraic. This implies that π is algebraic over $\mathbb{Q}$, but there is a famous Theorem of Lindemann (see [3], Chapter 6, for a proof) which asserts that π is not algebraic over $\mathbb{Q}$. Hence it is impossible to square a circle. $\square$

Neat, isn't it?

5.Epilogue

In the process of writing this article, I have discovered that I have only touched the tip of the iceberg, both in terms of geometric constructions and in terms of mathematical impossibilities.

As far as geometric construction goes, it has been known since the time of the Greeks that if one relaxed the condition on the use of ruler and compass alone, then these three problems could be solved. For example, Problem 1 was solved by the Greeks using a ruler in the form of a right angle (see p.146 of [6], for details) and using the *spiral of Archimedes* one can square a circle and trisect an angle (see pp. 95–96, of [1] for details). There are even variations with regard to the traditional ruler and compass as tools. If we agree that a line can be drawn if we are given two points on it, then it can be shown that all the ruler and compass constructions can be constructed using a compass alone (first discovered by Georg Mohr in 1672, and later discovered by and attributed to Lorenzo Mascheroni). It is also known that all ruler and compass constructions can be performed by using a two-edged ruler, whose edges are either parallel or meet at a point.

One of the reasons Gauss is revered so much is that at the age of 19, he demonstrated how to construct a regular polygon of 17 sides, using straight edge and compass! (See box item by Renuka Ravindran in *Resonance*, Vol.2, No.6, p.62) This after nearly 2000 years of stagnation as far as construction of regular polygons was concerned. Till then the Greeks had constructed regular polygons with 3 sides, 5 sides and in general regular polygons with 2^m sides, for $m \geq 2$. They also knew that if they could construct polygons with r sides and s sides, then they could construct one with side rs. Gauss showed further, that a regular polygon of n sides can be constructed whenever n is prime and is of the form $n = 2^k+1$ for some integer $k \geq 0$. These primes are actually called *Fermat primes.* You can check that 17 is one such. So from our discussion we see that we can construct a regular polygon of n sides provided it is of the form $n = 2^{i+2}$ or $n = 2^i p_1 p_2 \ldots p_j$, where $i \geq 0$ and $p_i, 1 \leq i \leq j$, are Fermat

Suggested Reading

[1] Howard Eves, *An Introduction to the history of Mathematics*, Holt, Rinehart and Winston, New York, 1976.

[2] William Dunham, *Journey Through Genius*, John Wiley and Sons, 1990.

[3] Ian Stewart, *Galois Theory*, Chapman and Hall, London, 1991.

[4] I N Herstein, *Topics in Algebra*, Blaisdell, New York,1964.

[5] Arthur Jones, A Sidney Morris and Kennneth R. Pearson, *Abstract Algebra and Famous Impossibilities*, Springer-Verlag, New York, 1991.

[6] Richard Courant and Herbert Robbins, *What is Mathematics?*, Oxford University Press, New York, 1941.

[7] M Rosenlicht, Integration in Finite Terms, *American Mathematical Monthly*,79,963–972, 1972.

[8] R Hadlock Charles, *Field Theory and its Classical Problems*, Carus Mathematical Monographs, No. 19, Mathematical Association of America, 1978.

primes. The question that naturally comes up is, what if n is not of this form. Well! Pierre Wintzel in 1837, proved that if n is not of this form, then it is **impossible** to construct a regular polygon with n sides.

The next famous impossibility comes from the area of solving polynomial equations with real coefficients. We all know the quadratic formula $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, where $a \neq 0$, is the solution to the equation $ax^2 + bx + c = 0$. If you notice the solution involves only the coefficients of the equation and square roots. Such a solution is referred to by the term *solutions by radicals.* That is, the solutions are obtained using field operations and extraction of roots alone. Once again it is natural to ask: is there a solution by radicals for a polynomial of arbitrary degree? The answer is yes if the degree is 3 or 4. But for degree 5, Niels Abel showed in 1824 that it is **impossible** to solve an arbitrary fifth degree equation by radicals alone. Just as in the case of the trisection of an angle, one should note that there may be some fifth degree equation, which does have a solution by radicals, but there is no general formula which takes care of all fifth degree equations. Actually the punchline to this story was given by Évariste Galois, who showed that in fact for an integer $n \geq 5$ it is **impossible** to solve an arbitrary equation of degree n by radicals. The tools used by Abel and Galois, form what is today known as Galois theory.

The last example we choose is from the area of calculus. We all know that no matter how hard we try we are unable to find $\int e^{x^2} dx$ in 'closed' form, i.e., in terms of functions usually defined in introductory calculus courses. Joseph Liouville proved that it is **impossible** to find $\int e^{x^2} dx$ in closed form. In [7] a more modern proof, based on the ideas of Galois theory has been provided.

So next time you come across the phrase 'nothing is impossible' don't believe it!

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