Music, Mathematics and Bach

1. Layers of Melody

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Music is the most abstract of the arts; it is not descriptive, it does not portray pictures or stories, and even interpretations of such things as 'moods' vary greatly across cultures. Nonetheless there are apparently things about this art which appeal strongly to the sort of people who are attracted to science. To take some well known examples, Einstein played the violin; Edward Teller was an accomplished pianist; the composer Borodin was a chemist by profession; C V Raman wrote papers on the acoustics of violins, and on Indian drums; the astronomer Herschel was also a composer and a conductor; and a number of other scientists have declared their fondness of music, as listeners if not as performers. In fact there is a lot of scientific basis to the practice of music, not only in such things as the design of musical instruments, auditoria, and (in the present day) recording and reproduction systems, but also in the nature and structure of music itself. Though what sounds melodious, or harmonious, or tuneful, must ultimately be regarded as a subjective matter, there seem to be physical justifications for the things in music which appeal aesthetically to most people. Besides, scientists are fond of patterns, and much of music consists of melodic and rhythmic patterns put together in an orderly, but creative manner.

Rather than dwell on the science of music generally, here we'll limit ourselves in the main to the music of one individual: Johann Sebastian Bach. This we do because the larger topic is a bit too large (Sir James Jeans, among many others, wrote an entire book on it, and even so he limited himself to Western classical music), because Bach does illustrate the above points nicely (and in discussing that we'll cover more general ground as well), and because the topic of Bach's music is a favourite of mine.

The music of Bach seems to have a particularly wide following among scientifically or mathematically minded people (and, conversely, dismissed sometimes as too 'intellectual' by others). One could point to several possible reasons for this – the highly regular structures, the complex counterpoint, the use of simple themes to construct elaborate structures – though, as with any other art form, it is impossible to adequately explain the appeal of Bach's music in words. One must listen to it, and if possible play it oneself. While Bach's music, like that of any great composer, does exhibit a full range of emotions such as joy, meditativeness, pathos, grandeur, awe, much of its appeal does seem to be cerebral rather than emotional.

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Moreover, Bach's career illustrates a rare example of a triumph of science over art. The system of 'equal temperament' in the tuning of keyboard instruments, of which he was one of the earliest and most active advocates, is mathematically elegant and of immense practical utility, but musically imperfect; thanks largely to Bach, the western musical world has lived with and thrived in this imperfection for over two centuries. Without equal temperament much of the piano music of the last two centuries would not have been written.

Melody and the Musical Scale

The chief components of music, it would almost universally be agreed, are melody and rhythm. To these the Western musician would add 'harmony'; and the concept does exist in Indian music, though not to the same extent as in the west. One can have music without some of these elements: for instance, an *alap* doesn't have rhythm, a percussion solo doesn't have melody, and much of 19th century Western music and later consists mainly of intricate harmonic changes without much evidence of either melody or rhythm. Nonetheless, rhythm is probably the most primitive concept, being present in our very bodies as well as in various phenomena in the natural world. One can have an entire concert based on rhythm (a *tala vadya kacheri*). Even an *alap*, which lacks a regular metre, has a temporal component to it: the

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speed, the pauses between notes, have to be right. Melody comes next, and consists of production of sounds of varying pitch. It is much less intuitively clear, however, what kinds of sequences of varying pitches constitute pleasing melodies. But though melodies vary greatly from culture to culture, some aspects of melody appear to be universal.

To begin with, every culture recognizes an interval called the 'octave'; given a note of a particular pitch, one can find a higher-pitched note, which sounds very similar. This note has twice the frequency of the lower note, and is generally called by the same name (such as 'sa'). One can recognize very early, too, that two notes separated in pitch by what is called a 'fifth' by Western musicians (that is, the interval between 'sa' and 'pa') seem to have a special relationship. (The name 'fifth' does not refer to any ratio, but arises from the fact that the 'pa' or 'panchama' is the fifth note in the scale, both in India and in the west.) It turns out that the higher note, in that case, has exactly 3/2 times the frequency of the lower note.

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Next, if one chooses a note a fifth above 'pa', one obtains a note of frequency 9/4 times that of the first. For convenience we can restrict our notes to the octave starting with the first note, and lower this note by an octave (that is, divide its frequency by 2) to bring it into this range. If one repeats this process twice more, one ends up with six notes (including the higher 'sa') of the following frequencies:

1, 9/8, 81/64, 3/2, 27/16, 2

This exercise is not merely academic: these turn out to be close to the pitches of the five notes in what is called the 'pentatonic scale', which is a very commonly used scale in many different cultures around the world – indeed, it is the earliest known scale in common use. It is the scale of Raga Mohanam (in Carnatic music) or Bhopali (in Hindustani music). (Another curiosity is that if one chooses the same set of pitches but starts from the third, whose pitch above is 81/64, one obtains the scale of Malkauns or Hindolam.)

One can add two more notes to the same scale, and the result is very close to the seven-note scale used as the basic scale in music all over the world. If one simply follows the same procedure as above, one gets the eight notes (including the final octave):

1, 9/8, 81/64, 729/512, 3/2, 27/16, 243/128, 2

When one plays this, it sounds very wrong: the fourth note is too high. (We would call it '*tivra ma*'.) So one chooses the 'fourth' (the 'shuddhama') to be a fifth below the octave, that is 2/3 of the octave, so one has

1, 9/8, 81/64, 4/3, 3/2, 27/16, 243/128, 2

as the frequencies of the eight notes in the scale. In other words, this scale can be generated by starting from the fourth note and going up successive fifths (that is, multiplying its frequency by 3/2 and dividing when necessary by 2 to keep it within the octave). This is called the Pythagorean scale; the reader can program a computer to generate sounds of these frequencies, and it is essentially the scale most commonly used in music, the scale of *Raga Shankarabharanam*, known in western music as the major scale.

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In fact, the first and the third notes in the above Pythagorean scale also seem to have a certain close relationship and sound pleasant when played together, or successively. It was known in the ancient world that certain 'intervals' – that is, differences in pitch between notes – are 'consonant' (sound pleasant to the ear) and certain other intervals are 'dissonant' and sound jarring. Pythagoras made a careful study and concluded that sounds whose frequencies are in small integer ratios are consonant. The notion of frequency wasn't so clear at that time, of course, so his statement was in terms of lengths of vibrating strings under uniform tension. Thus the 'fifth', the interval between 'sa' and 'pa', being in the ratio 3/2, is consonant; but the 'sa' and 'ri', being in the ratio 9/8, don't sound particularly pleasant when sounded together. The 'third', or the interval between 'sa' and 'ga', is written above as 81/64 which is very nearly 5/4; to improve the Pythagorean scale we could therefore make it exactly 5/4 (which makes the interval between this note and the fifth 6/5).

Why small integer ratios should be pleasant is not universally agreed upon either, but there are some likely explanations. To start with, a sound which sounds musical (has a definite pitch) has a periodic waveform. Such a waveform can be decomposed (Fourier analyzed)[1] into sine waves of various frequencies and the higher frequencies (called overtones or harmonics) being integer multiples of the lowest, which is called the fundamental. (This was implicitly assumed above in talking about frequencies of notes: we meant the fundamental frequencies.) Now, a superposition of notes with small-integer-ratio frequencies will also have a periodic waveform with a small period, so presumably it sounds musical compared to the superposition of two arbitrary notes. Some evidence for this is the observation that when one superposes notes with frequencies 2, 3, 4 ... times a 'missing' fundamental frequency, to generate a waveform with that fundamental frequency, the ear often perceives it as having the missing fundamental frequency which in fact does not appear in the Fourier analysis of the sound. This is also how a portable cassette player with small speakers can produce the effect of bass sounds, though sounds with those actual frequencies cannot really be reproduced on such small speakers: the speakers reproduce the overtones and the ear (or mind) tends to 'fill in' the missing fundamental.

[1] See S Thangavelu, *Resonance*, Vol.1, No.10, p.44, 1996.

Another point is that when notes in small integer ratios are played together, they will have harmonics in common (for instance, the third harmonic of C or 'sa' is the second of G or 'pa') and these tend to reinforce each other and, presumably, have a pleasant effect. Otherwise one has an arbitrary distribution of harmonics, some of which have no relation at all, while others may accidentally lie somewhat close in frequency and cause 'beats' which can be unpleasant.

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Whatever the reason, it seems that notes in small integer ratios sound good when played together and this should influence the structure of the scale. Although in Indian music notes are seldom played together on a particular instrument (and with a single voice it is an impossibility in any case), all notes are played

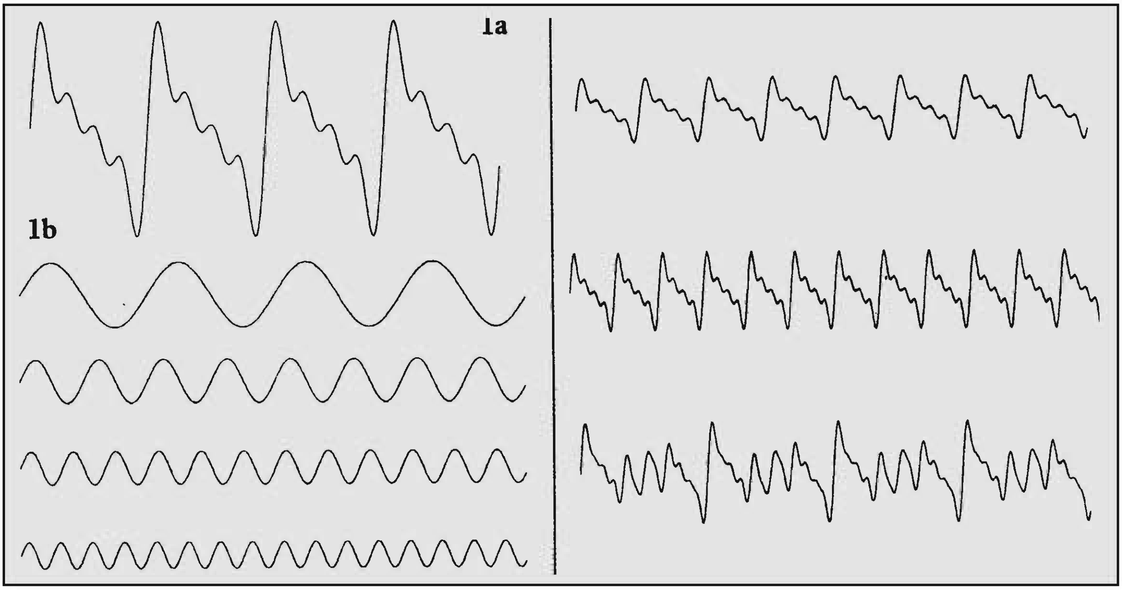

Figure 1 (left). A periodic waveform emitted by a musical instrument, such as 1a which is similar to what is produced by a violin is really composed of a lot of sine (or cosine) waves whose frequencies are multiples of the 'fundamental' frequency of the waveform.

Figure 2(right). When two sounds separated by a 'fifth' are combined (the middle waveform above has 3/2 times the frequency of the upper waveform), the result (bottom) is also a periodic wave with a reasonably small period. This is true whenever sounds with frequencies in small integer ratios are combined.

against the constant drone of the tanpura, so their harmonic agreement should certainly be a consideration.

The ancient Greeks had in fact several scales or 'modes', which depended on which note one started from when constructing the scale as above in a succession of 'fifths'. These modes survived well into medieval times, as long as melodies were essentially 'linear' – one note at a time (as in Indian music). Eventually, however, all but two more or less died out: the major scale described above, and the minor scale to be described later.

Layers of Melody: Counterpoint and Harmony

Around the twelfth century a new development took place in Western music, which would greatly influence its further evolution and take it in a direction different from the music of

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most other cultures. The idea of 'part singing' evolved, where different voices in a vocal ensemble sing different melodies, which in general would be a very jarring thing, but these melodies are chosen and combined in such a way as to have a pleasant overall effect. How they should be combined depends crucially on the ideas of consonant intervals discussed above: the memories should interweave in such a way that whenever two notes are sounded together, they should be consonant, or at least this should happen at the important points. There is much more to it than that: the melodies should be related rhythmically, and in other subtle ways, but at the same time they should give the impression of being free and independent, not artificially contrived to match each other.

At the simplest level, then, there could be two voices, one singing the main melody and the other an accompaniment which is a different tune but harmonically related to the first. Some Christmas carols such as 'Silent Night' are occasionally sung in this manner. One could add a third voice, and a fourth, and so on, but to do this while maintaining the smoothness and continuity of each as well as the harmonic integrity of the whole is a very challenging task. On the other hand, when well executed the effect is extremely striking. This kind of music is called 'polyphony' or 'counterpoint', while one-voice music is called 'monophony'. The word 'voice', as used here, refers to an individual melodic line which could be instrumental as well as vocal.

This had two important effects on western music: first, it added the 'vertical dimension' to music, the superposition of different sounds on top of each other; second, it gave composed, written music a dominant position over spontaneous, improvised music, since it is extremely difficult for two people to effectively improvise counterpoint. (It is still possible for one skilful person, playing an instrument such as the lute or a keyboard instrument, to improvise counterpoint, and this was done quite frequently. But written music also survives longer: the spontaneous improvisations of the Baroque musicians are lost to us.) Through the middle ages and the Renaissance, the complexity and sophistication of contrapuntal music grew, and some of the music for small ensembles of instruments or voices survives in written form to this day.

Around the 1600s another development took place, started by composers of what is called the 'Baroque' period which is the earliest period of 'classical' western music. As a reaction to the complexity of contrapuntal music and the excessive mental concentration required of the listener to follow it, a return took place to music having a single, dominant melodic line. Now, however, instead of being played on a single voice or instrument or on an ensemble all playing the same note, each note of the melody was reinforced by harmonically related notes from below or above. This kind of writing, in contrast to monophony or polyphony, is called 'homophony'. A group of harmonically related notes played together is called a 'chord': thus, at the opposite extreme from polyphony, a composition could now become a single melody reinforced by a background accompaniment of chords, which is how much Western popular music is composed even now.

Another reason for this trend was the development of opera, with vocal parts that told a story (through dialogues and so on, which were sung rather than recited): the new form made it easier to understand the words.

Actually there is no sharp dividing line between the two: a harmony to a melody is a set of notes accompanying the melody and sounding consonant with the melody, and this accompaniment can be regarded as a harmony or as an independent contrapuntal 'voice' depending on its sophistication or its own melodic quality. Much Baroque music retained some contrapuntal elements. These nearly disappeared in the early *classical* period (Bach's sons, or the young Mozart and Haydn) but reappeared soon after. Indeed, most Western music has both harmonic and contrapuntal aspects; but at this point, rather than explore this further, we will examine Bach's music a little more closely in the next part of this article.

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