momentum and position rather than numbers. Schrödinger's equation followed soon after, and was a partial differential equation whose solutions yielded the same energy levels as Heisenberg's matrices. At that time this seemed strange, but Schrödinger soon set his method into the more general framework of vector spaces and showed that his method and Heisenberg's were really the same. Born, Dirac, and many others contributed to this step. So while the uncertainty principle relating to position and momentum was first suggested by Heisenberg and is known by his name, in a more general context such a principle follows from the work of several other people, including Schrödinger, whom this issue commemorates. The final structure of quantum mechanics emerged from the collective efforts of various people including Heisenberg and Schrödinger, and from this structure various generalizations of the uncertainty principle can be derived.

The Three Colour Problem

Dinoj Surendran
12, Langham Road, University of Zimbabwe, P.O. Box. MP 167, Harare, Zimbabwe.

One of the most famous problems in the world is the *four colour problem*. This merely states a fact that any six-year old armed with crayons has long suspected – it is possible to colour any map in the family atlas with only four colours so that no two neighbouring regions have the same colour.

Mathematicians have this habit of being precise, and they define a map to be a partition of some finite area into finitely many contiguous regions. The contiguous bit merely stresses the point that each region must be connected, so countries like Angola which are composed of two different parts (the main part of Angola and the Cabinda enclave) are considered as two different regions. Another technical point to note is that two countries which only touch at a point (like Zimbabwe and Namibia) are not considered to be neighbouring.

Another problem with mathematicians (in the last century anyway) is that they tend to wait for someone to state a problem formally before they have a go at it. With the four colour problem that someone was Francis Guthrie, a South African student studying in London. But while he suggested it in 1852, it was only in 1878 that it attracted any interest – for that was when the well-known mathematician Arthur Cayley mentioned it at a conference. (Finicky bunch, these mathies!)

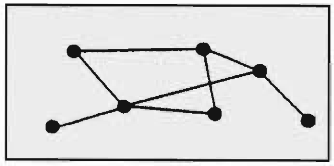
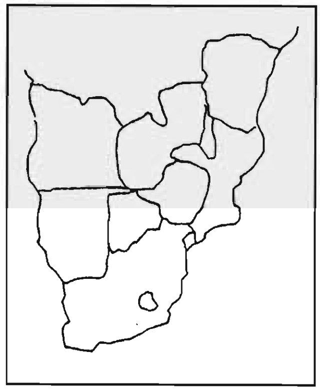


Figure 1.

Naturally, the problem seemed easy at first sight. But after a few proofs of it turned out to have flaws, researchers gave it a bit more respect. So there was great excitement in 1976 when Kenneth Appel and Wolfgang Haken announced a proof of it! But the cheers turned to jeers when they added that they had been forced to use a **computer** to check the thousands of possible cases of the problem. Times have changed, and computer-aided proofs are now considered acceptable. Still, the search continues for a shorter proof. In 1996, Neil Robertson, Daniel Sanders, Paul Seymour and Robin Thomas found just that – but it still required computer time.

Figure 2.

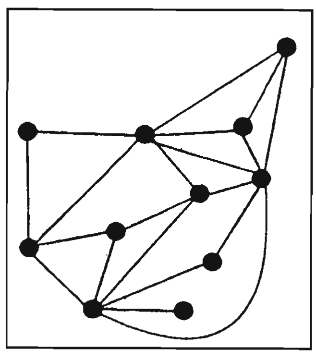


Graph Theory

There are several branches in mathematics, with new ones being created all the time. The branch dealing with this problem is called **graph theory**. But the graphs it deals with have nothing to do with x and y axes! They are collections of dots and lines (called vertices and edges respectively) and are subjected to the following rules:

1. Every edge joins two different vertices.
2. Every two vertices are joined by at most one edge.

Figure 3.



These rules are modified for different areas of graph theory, but will do for our purposes. This is an example of a graph. (*Figure 1*)

Graphs and Maps

It is easy to create a graph from a map – represent each region by a vertex and join two vertices if and only if their corresponding regions are neighbouring with quite a dubious map of Southern Africa (*Figures 2 and 3*). Next we **colour** the graph. This simply means we assign a colour (usually represented by a number) to each vertex so that no two neighbouring vertices (vertices with an edge between them) have the same colour. This clearly corresponds to a colouring of the map.

Graphs created in this way are called **planar** graphs. They correspond precisely to those graphs that can be drawn on a plane piece of paper so that no edges cross each other. Not all graphs are planar, for example the graph consisting of five vertices all joined to each other, but we shall restrict our attention to those that are.

As we allow ourselves more and more colours, it is easier to colour the graph. So to make life interesting, we must find the fewest number of colours needed to colour a graph. This number is called the **chromatic number** of the graph. We can now state the four colour theorem more formally:

"The chromatic number of any planar graph is at most 4."

The Three Colour Problem

But what we have said so far is no doubt familiar to many readers. What is less known is the *three colour problem*:

"What planar graphs can be coloured with only three colours?"

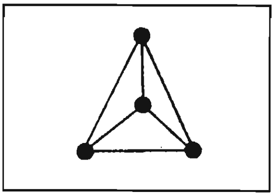


Figure 4.

It is easy to produce one which requires four colours – consider the graph of four vertices (*Figure 4*) all joined to each other. No two vertices can have the same colour since all are neighbouring to each other. There are four vertices, so at least four colours are needed.

In 1963, B Grünbaum proved that if a graph contained at most three triangles (three vertices joined to each other) then it could be coloured with three colours. Wait a minute! Haven't we just stated that the above graph needs four colours? Yes, but that graph contains *four* triangles, not three. If we label its vertices as *a, b, c, d* then *abc, abd, acd, bcd* are all triangles.

So Grünbaum's result is the best possible, which would seem to end the problem, but actually we've only just started! Because

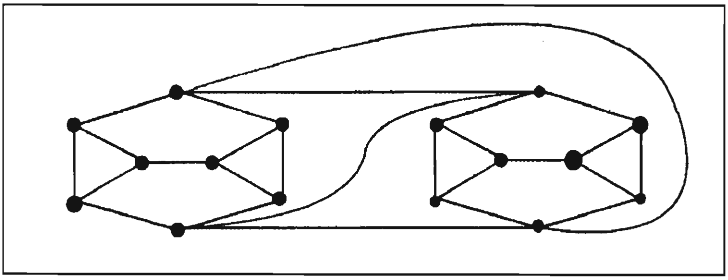


Figure 5.

intuition tells us that the further apart triangles are in a graph, the easier it will be to 3-colour it. Can we state this more formally? Let's define a new concept for a graph with at least two triangles – let d be the least number of edges one needs to travel on to get from some vertex of one triangle to some vertex of another triangle.

A Tale of Two Conjectures

Now, most textbooks present mathematics as a list of theorems which follow each other like mindless donkeys in a desert caravan. But the subject is really a lot more than that – for, behind every finished theorem there are several false starts in the form of wrong conjectures. I can think of a couple of reasons why we should look at these seemingly useless objects. Firstly, they can be quite interesting, especially if they collapse spectacularly! Secondly, they can open up new areas for research. The d we have just defined is involved in a very interesting story. For, Grünbaum made the following conjecture:

"If G is a planar graph for which $d \geq 1$, then G can be coloured with three colours."

Very nice in all respects except one – it was wrong! For, six years later, in 1960, Havel gave the counter example in *Figure 5*.

As you can see, this a planar graph for which $d = 1$. It is four-colourable but not 3-colourable, which we shall prove later on. Anyway, Havel decided that the conjecture was too good to simply throw out, and thus only slightly modified it: If G is a planar graph for which $d \geq 2$, then G can be coloured with three colours. Then – oops! The following year he disproved his own

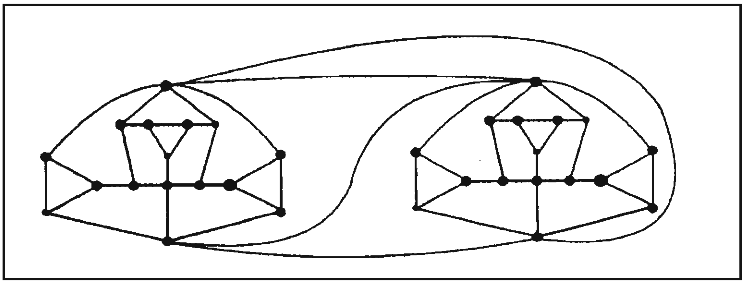


Figure 6.

conjecture with a counter example (*Figure 6*).

If this is beginning to sound like a tale of two conjectures, that's entirely possible, for there was a theorem next. But four more years had to pass – year, year, year, hear ye, Aksionov proved that if G is a planar graph for which $d \geq 3$, then G can be coloured with three colours. That was in 1974. At last! The problem was settled. Till 1980, when he and Melnikov found that his proof was leaking – so badly that they found another counter example! (*Figure 7*).

The Secret Behind the Counter Examples

Don't the three counter examples shown have something in common? Yes! They are all composed of two copies of a graph joined somehow. This graph is called, rather misleadingly, a *quasi-edge*. The special property of such a graph is that it contains two vertices not joined by an edge which must be coloured differently in any 3-colouring of it. For instance, in the quasi-edge of Havel's first counter example, the vertices v_1, v_8 will always have to be coloured differently (*Figure 8*).

This can be proved by contradiction. Suppose we can colour the quasi-edge with 1, 2, 3 so that v_1, v_8 are coloured 1. Then v_2, v_6

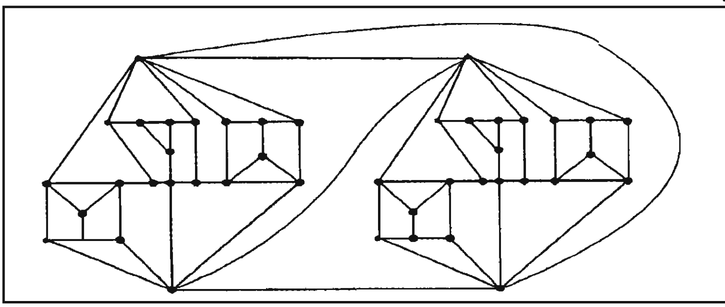


Figure 7 (left).

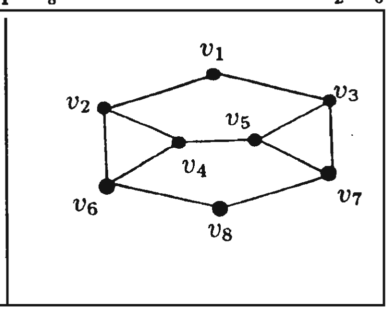


Figure 8 (right).

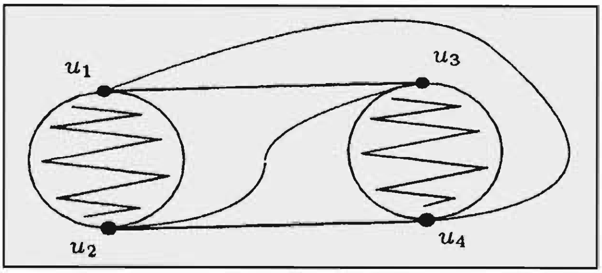


Figure 9.

will have two different colours, say 2 and 3 respectively. Ditto for v_3, v_7. This means that both v_4, v_5 will be have to be coloured 1, which cannot be. Q.E.D.

Now we can create a graph with chromatic number 4 from the quasi-edges (*Figure* 9).

Get two copies of the quasi-edge and call its special vertices u_1, u_2, u_3, u_4. Join them as shown. If u_1 is coloured 1, then u_2 must have a different colour 2. u_3 is neighbouring to u_1, so u_2 must have a new colour 3. Similarly u_4 cannot be coloured 1 or 2, nor 3 (because of the quasi-edge) so it requires a fourth colour.

What now?

For obvious reasons, mathematicians are being extra-careful in making conjectures in the direction of the above ones. Does there even exist a positive integer N such that any planar graph with $d \geq N$ is colourable with 4 colours? No one knows, but Aksionov and Melnikov have cautiously suggested that N exists and is 5. Given what we have just seen, that would seem to carry just about as much weight as a suitcase made of paper. Whatever the case, this problem is still open as far as I know.

There are however other results (true ones!) which tell you when a planar graph is 3-colourable. For instance, B Walls has recently proved that any planar graph that does not contain n-circuits, $r \leq n \leq 8$, is 3-colourable, and it is suspected that the result holds if the 8 is replaced by 5. An n-circuit is simply n vertices joined in a cycle, like a n-gon.

The author would like to thank Daniel Sanders for informing him of Walls' result, which is yet to be published.

Suggested Reading

[1] Oystein Ore. *The Four Colour Problem*. Academic Press. New York, 1967.

[2] L S Melnikov, V A Aksionov. Some counter-examples associated with the 3-colour problem. *Journal of Combinatorial Theory*. B28. 1–9, 1980.