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## The Uncertainty Principle for Dummies

I couldn't resist the choice of title, but in fact I am not supposing that the readers are dummies, I assume no prior knowledge of quantum mechanics, but a basic knowledge of vectors and of matrices would be useful. Readers who are unfamiliar with these can try ploughing ahead anyway, and they can look up a definition of matrix multiplication and play around the examples when they get that far.

The uncertainty principle of Heisenberg is one of the most famous statements in science this century, but it causes a lot of confusion among students and even among teachers. For instance, this periodical has been receiving requests to clarify issues like 'Is the uncertainty principle compatible with Schrödinger's equation?' 'Where does the uncertainty principle come from?' and 'Must any solution of Schrödinger's equation obey the uncertainty principle?'

In fact the uncertainty principle has nothing to do with Schrödinger's equation. It comes from the basic assumptions of quantum mechanics: it must automatically be obeyed by *any* state, *whether or not it obeys the Schrödinger equation*. In addition, of course, any state given at an instant as a function of spatial coordinates must evolve in time according to Schrödinger's time dependent equation.

The confusion perhaps stems from the usual statement of the uncertainty principle in terms of position and momentum, and one's confusion with the classical versions of these objects which always have exact values. The student can't see why they don't (and cannot) have exact values in quantum mechanics (at least not simultaneously). The reason is that in quantum mechanics, these objects are not numbers, but 'operators' which act on a state, and may either leave it unchanged but multiplied by a constant (so that they're effectively like numbers) or change it entirely to something else (so that they're not like numbers at all). Our classical idea of a position, or a momentum, is then

some sort of averaging of the real effects of these operators.

All this tends to be a bit too confusing for the beginning student so we'll try to clarify it with the example of spins, which have their own angular-momentum uncertainty relations, and which are also not a comfortable topic for beginners. After hopefully making this a bit clear, we'll move on to the case of position and momentum. We won't actually derive the uncertainty principle, but merely make it plausible.

### Basic Postulates of Quantum Mechanics

The following are the basic ideas of quantum mechanics. These are generally intimidating to beginners, so I'll try to explain them using the specific example of a spin system. If the reader gets comfortable with the idea of an observable (like position or momentum or spin) being an operator, rather than a number (or, more likely, gives up trying to understand and decides to take it on faith), then the rest is not difficult.

Each state of a system is represented by a vector of unit length in a Hilbert space.

We won't get into the definition of a Hilbert space here; but our ordinary three dimensional space, with a vector being the triad of coordinates of a point, is an example (though a very simple one). More generally, so is an  $N$  dimensional vector space similar to our three dimensional one. In quantum mechanics, the Hilbert spaces we need are usually infinite dimensional, and always complex – that is, the components of vectors are complex numbers, and we can multiply them by complex numbers.

For instance, if the system is a particle whose spatial coordinates we are ignoring, and whose intrinsic angular momentum (or 'spin') can point in either the  $+z$  and  $-z$  direction, its states can be  $(1, 0)$  (spin up),  $(0, 1)$  (spin down), or any other linear combination of these with complex coefficients whose squared sum is unity: that is, a general two component complex vector with unit magnitude.



An essential difference of these states from classical states is the idea of 'superpositions of states' – classically, a spin must point in one direction or another, but quantum mechanically both possibilities can exist, which is why we need a two component vector to describe the states rather than an ordinary number.

If the system is a single particle in one dimension, the 'vector' could be a complex-valued function of the coordinate. It could also be a complex-valued function of the momentum, or any other observable. The choice is ours, but in most elementary problems one chooses the coordinate. You can think of the wavefunction,  $\psi(x)$ , as a 'vector' whose index is continuous ( $x$  varying in an interval on the real axis), rather than discrete as in the usual vectors  $\psi_n$  where  $n = 1, 2, 3$  for instance.

All observables of a system are represented by linear Hermitian operators acting on the Hilbert space, and their allowed values are the eigenvalues of these operators. For each observable, the corresponding eigenvectors form a complete set which spans the Hilbert space.

An operator is just something which transforms a vector into another vector. Linear means that when it acts on the sum of two vectors the result is the same as if it acted on each individually and then you added the transformed vectors. An eigenvector of the operator is a vector which is unchanged apart from a multiplicative factor when the operator acts on it. An eigenvalue is the corresponding multiplicative factor. An  $N$  dimensional linear operator has  $N$  eigenvalues. For a Hermitian operator all the eigenvalues are real.

For instance, in the two-component spin case above, the  $z$  component of the spin could be represented by a  $2 \times 2$  matrix, and so could the  $x$  and  $y$  components. These are linear operators, and their operation on the state is a matrix multiplication into the two-component vector which represents the state. An eigenvector of the operator is a vector which, when acted on by the operator, merely gets multiplied by a number. That number is called an eigenvalue.

To take a very simple example, if the  $z$  component of the spin,  $S_z$ , is taken to be

$$S_z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

the eigenvectors are

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

with eigenvalues  $1/2$  and  $-1/2$ , respectively.

If a state is an eigenstate of an operator, any measurement of the corresponding observable will always give the corresponding eigenvalue of that operator. If a state is a superposition of eigenstates, the observation will pick one of the eigenvalues according to the absolute square of the weight of the corresponding eigenstate in the initial state of the system, and the system will 'collapse' to that eigenstate.

For instance, if the state of the above two-component spin is  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ , a measurement of the  $z$  component of the spin will always yield the value  $+1/2$ , whereas if the initial state is  $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ , the measurement will yield  $+1/2$  or  $-1/2$  with equal probability. Generally, if  $\psi_1, \psi_2, \psi_3, \dots$  are the eigenstates of the operator, and the system is in a state  $\Psi = \alpha_1 \psi_1 + \alpha_2 \psi_2 + \dots$ , where  $\alpha_1, \alpha_2, \dots$  are complex numbers whose absolute squares sum to 1, then the probability that a measurement will find the system in the state  $\psi_1$  is  $|\alpha_1|^2$  and so on.

These are the postulates of quantum mechanics, and only the dynamics remains to be described: that's where Schrödinger's equation comes in. That equation describes how a state evolves with time. But we won't get into that, but rather describe how the above postulates, without Schrödinger's equation, lead to the uncertainty principle.

## Spins

Though the above is not as hard as it seems at first sight, it

requires a new way of thinking about the system. To get the reader used to this, let us discuss the intrinsic angular momenta, or spins, of quantum mechanical particles.

Angular momentum, like linear momentum and position, does not mean quite the same thing in quantum mechanics as it does in classical mechanics. It becomes the same thing in the limit of large quantum numbers – but while the ‘orbital’ angular momentum can become as large as you like, the intrinsic ‘spin’ is fixed, and therefore there is no classical analogue of it. The spin of an electron is not its rotation about an axis. It is a thing in itself, without a classical counterpart.

Basically, however, quantum mechanical angular momentum has three components, each represented by an operator. In addition, the square of the total angular momentum is the sum of the squares of these operators. These operators obey certain ‘commutation relations’ which we’ll come to soon, and which imply that the squared total angular momentum,  $\mathcal{J}^2$ , can only have eigenvalues of the form  $\hbar j(j+1)$  where  $j$  is an integer or half an odd integer; and the  $z$  component of the angular momentum,  $\mathcal{J}_z$ , can only have values  $m = j, j-1, \dots, -j$ . There’s nothing special about  $\mathcal{J}_z$ : this is true of  $\mathcal{J}_x$  and  $\mathcal{J}_y$ , too, and one can pick axes in any direction one chooses, and this is still true. It sounds counterintuitive if one thinks of an angular momentum as a classical vector whose  $z$  component must vary continuously as one turns the axes. But this angular momentum is not a classical vector. It is a quantum mechanical vector operator, and no matter how you pick your axes, the  $z$  eigenvalue must be an integer or half an odd integer (we call the latter a half-integer for short). The total squared angular momentum must be  $\hbar j(j+1)$  where  $j$  is the maximum allowed value of the  $z$  component. For simplicity, in this section we will put  $\hbar = 1$ .

Let us take the smallest value allowed for  $j$ , namely  $1/2$ . The allowed values of  $S_z$  are  $\pm 1/2$ , and the total spin is  $S^2 = (1/2)(1/2 + 1) = 3/4$ . We can represent a state of the system by

$$\psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

denoting respectively an up spin or a down spin; a state can also be a linear combination of these states

$$\psi = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where  $\alpha$  and  $\beta$  are complex numbers and  $|\alpha|^2 + |\beta|^2 = 1$ .

The most general linear operator on these states would be a  $2 \times 2$  matrix which multiplies these and transforms them into something else. If we want the states representing up and down spins to be eigenstates of  $S_z$ ,  $S_z$  must be a diagonal  $2 \times 2$  matrix; it is easy to see that

$$S_z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenvalues of this matrix are  $\pm 1/2$ , corresponding to the up and down states above.

Now what are the matrices representing  $S_x$  and  $S_y$ ? It can be shown that the three matrices representing  $S_x$ ,  $S_y$  and  $S_z$  should satisfy the commutation relations

$$\begin{aligned} S_x S_y - S_y S_x &= i S_z \\ S_y S_z - S_z S_y &= i S_x \\ S_z S_x - S_x S_z &= i S_y \end{aligned}$$

and the total spin  $S^2$  should commute with all the components ( $S^2 S_x - S_x S^2 = 0$ , etc). This is satisfied if we take

$$S_x = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, S_y = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

There is nothing special about choosing eigenstates of  $S_z$  as our basis: we could equally choose

$$\psi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

which are eigenstates of  $S_x$  with eigenvalues  $1/2$  and  $-1/2$ , respectively; or

$$\psi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \text{ and } \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

which are eigenstates of  $S_y$ . Or we could simply choose different matrices (perhaps permutations of the above) for  $S_x$ ,  $S_y$ ,  $S_z$  keeping their commutation relations and eigenvalues the same. The above choice is the conventional one, and these matrices are called the Pauli spin matrices.

The point to be noted is this: The eigenstates of  $S_x$  are not the eigenstates of  $S_z$ , but a linear combination of them. So a state which has a definite value of  $S_z$  does not have a definite value of  $S_x$  and vice versa. This is an uncertainty principle for the components of spin (angular momentum).

Most generally, an uncertainty principle arises whenever the operators corresponding to two observables don't commute. If two operators commute, one can always find simultaneous eigenfunctions for them, so one can have states with definite values for both observables. But if they do not commute, one cannot in general find states which are simultaneous eigenfunctions of both, and we must have uncertainty.

Supposing we had a hypothetical state  $\psi$  which was an eigenstate of  $S_z$ , with eigenvalue  $s_z$  and suppose the same state were an eigenstate of  $S_x$  with eigenvalue  $s_x$ . Then consider the operation of the commutator of  $S_z$  and  $S_x$  on  $\psi$ :

$$(S_z S_x - S_x S_z) \psi = S_z (s_x \psi) - S_x (s_z \psi) = s_x S_z \psi - s_z S_x \psi = s_x s_z \psi - s_z s_x \psi = 0$$

since  $s_x$  and  $s_z$  are ordinary numbers which commute with the operators  $S_x$  and  $S_z$ . But we also know that



$$S_z S_x - S_x S_z = i S_y.$$

So unless the eigenvalue of  $S_y$  for this state of zero, such a state cannot exist. In the spin half case we were discussing above, the eigenvalues of  $S_y$  were  $\pm 1/2$ , not zero; so no state can simultaneously have exact values for  $S_z$  and  $S_x$ .

Thus, the uncertainty principle really arises not from imperfect measuring devices or limitations in our understanding, nor from the dynamics as described by Schrödinger's equation: it arises from the very structure of quantum mechanics itself. When we state that observables are not pure numbers but operators, and the observed values of these observables are the eigenvalues of these operators, that alone is sufficient to ensure that two operators which do not commute must be related by an uncertainty principle.

There is much more to the spin matrices and the commutation relations than has been described here. In fact the allowed eigenvalues of the angular momentum can be deduced entirely from the commutation relations. Moreover, there are simple ways to work out the analogues of the Pauli matrices for systems with higher spin. These topics are discussed in several textbooks, so we don't pursue them further here.

### Position and Momentum

The same thing happens with position and momentum: the operators  $X$  and  $P_x$  (the position and the momentum) do not commute. Their commutator is given by

$$XP_x - P_x X = i\hbar$$

This is a fundamental postulate of quantum mechanics; it is not provable. (In a sense, the angular momentum commutation relations are also postulates: in the case of orbital angular momentum, they can be derived from the above position-momentum uncertainty relation, but spin angular momentum is a thing in itself and not derivable from the above, so we must



take the commutation relation there as an assumption.)

Note that here both  $X$  and  $P_x$  are operators. What these operators look like depends on how we are describing our system; most commonly, we simply describe it by a wave function which is a function of the position  $x$ . Then we can take the position operator  $X$  to be just the position  $x$  (a pure number) and the momentum operator to be  $-i\hbar \partial/\partial x$ , which is a differential operator acting on the wave function, not a number. The reader can verify that this choice satisfies the commutation relation.

Now where does the uncertainty principle come in? It arises from the fact that  $P_x$  and  $X$  don't commute. So we can immediately see, as above, that a state with definite values of  $P_x$  and  $X$  cannot exist. If it did, the value of  $(XP_x - P_x X)$  acting on this state would be zero: but it cannot be zero; it must be  $i\hbar$  times the original state (no matter what state it acts on).

The representation of the position and momentum operators above is useful in understanding what's going on. In this representation, a position eigenstate is a wave function  $\psi(x)$  which is sharply peaked at one point and vanishes everywhere else, but whose integral over all space is 1. This is called the Dirac delta function, and requires some sophisticated arguments to strictly justify—the justification was not made till the 1960s but physicists freely used it everywhere after Dirac introduced it in the 1930s. This is obviously an eigenstate of the 'operator'  $X = x$  because  $x\psi(x) = x_0\psi(x)$ ,  $x_0$  being the only point where  $\psi(x)$  is nonzero. To apply  $P$  on it is a bit tricky, and the best way is to take  $\psi(x)$  as a limit of a high, narrow wave packet. But one can convince oneself quite easily that  $\psi(x)$  is not an eigenstate of  $P$ .

$P$  being a differentiation operator, it is clear that its eigenstates (functions that stay the same apart from a multiplicative constant when acted on by  $P$ ) will be exponentials. If  $P$  is Hermitian (has real eigenvalues) the exponent will be complex, so the eigenstate would look like  $\exp(ikx)$  ( $= \cos kx + i \sin kx$ ). This is an oscillating function with wavelength  $2\pi/k$ , but it's oscillating not in



amplitude but in the complex 'phase': its absolute value is the same everywhere. So the momentum eigenstate is completely uncertain in position, and the position eigenstate (it can be shown) is completely uncertain in momentum. The momentum eigenvalue is  $\hbar k$ .

To really follow these ideas to the fullest extent, it would be good for the reader to be familiar with Fourier series and Fourier transforms. The 'momentum space' representation, and the uncertainty principle in this case is really an example of a well-known uncertainty relation in wave physics between the 'spread' of a wave packet and the 'spread' of its constituent wavelengths (Fourier transform). Some of this ground, and related 'uncertainty principles' in mathematics, are discussed elsewhere in this issue.

Even if the reader is not familiar with Fourier transforms, one can convince oneself that a large number of waves with nearly equal wavelengths can form a 'wave packet': there will then be a place where (by design, or coincidence) they all add up constructively to give a finite value, but as we move away from that point, their amplitudes become more and more uncorrelated since their wavelengths are all different, and so the total amplitude falls. To get a narrower wave packet, we need a bigger spread in wavelengths for more effective cancellation away from the maximum.

## Conclusion

Above, we have treated two special cases of the uncertainty principle. The principle is much more widely applicable than that, but these two examples serve to convey the flavour of what is going on.

Historically, the development of quantum mechanics was not quite so simple as above, and the postulates which we made in the second section developed quite gradually. The first version of quantum mechanics was Heisenberg's 'matrix mechanics', where he treated some specific problems using matrices for



momentum and position rather than numbers. Schrödinger's equation followed soon after, and was a partial differential equation whose solutions yielded the same energy levels as Heisenberg's matrices. At that time this seemed strange, but Schrödinger soon set his method into the more general framework of vector spaces and showed that his method and Heisenberg's were really the same. Born, Dirac, and many others contributed to this step. So while the uncertainty principle relating to position and momentum was first suggested by Heisenberg and is known by his name, in a more general context such a principle follows from the work of several other people, including Schrödinger, whom this issue commemorates. The final structure of quantum mechanics emerged from the collective efforts of various people including Heisenberg and Schrödinger, and from this structure various generalizations of the uncertainty principle can be derived.

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### The Three Colour Problem

One of the most famous problems in the world is the *four colour problem*. This merely states a fact that any six-year old armed with crayons has long suspected – it is possible to colour any map in the family atlas with only four colours so that no two neighbouring regions have the same colour.

Mathematicians have this habit of being precise, and they define a map to be a partition of some finite area into finitely many contiguous regions. The contiguous bit merely stresses the point that each region must be connected, so countries like Angola which are composed of two different parts (the main part of Angola and the Cabinda enclave) are considered as two different regions. Another technical point to note is that two countries which only touch at a point (like Zimbabwe and Namibia) are not considered to be neighbouring.

Another problem with mathematicians (in the last century anyway) is that they tend to wait for someone to state a problem formally before they have a go at it. With the four colour

