Uncertainty in the Real World

Part 1: Fuzzy Sets

Satish Kumar

"Think of arm chairs and reading chairs and dining-room chairs, and kitchen chairs, chairs that pass into benches, ... settees, dentist's chairs, thrones, opera stalls, seats of all sorts ... and you will see what a lax bundle in fact is this simple straightforward term. I would undertake to defeat any definition of chair or chairishness that you gave me."

Charles Pierce
Dictionary of Philosophy and Psychology
Volume 2, 1902

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In this two part article, Part 1 introduces the concept of a fuzzy set as opposed to that of a classical set. The notion of fuzziness as a means of modelling linguistic uncertainty is developed through examples. Elementary operations of fuzzy sets are introduced and are related to a geometry of fuzzy sets. This will develop a base for Part 2 in which we will understand how fuzzy systems work.

Fuzzy Uncertainty

Uncertainty pervades our every day lives. From stock market index fluctuations to weather prediction and traffic control, in almost every domain to which we turn our attention, we come face to face with uncertainty. Uncertainty can be of different kinds. For example, it may be probabilistic in nature, as in the case when we toss a coin where there is an uncertainty of whether the outcome will be a *head* or a *tail*. Here, the uncertainty vanishes *after the event occurs* – the outcome must be either a head or a tail. There is no ambiguity. Either the coin toss is a head or it is not. This is an example of *classical bivalence*.

Alternatively, uncertainty may be *fuzzy*. For example, if we say

There has been much debate over whether fuzziness is indeed different from probability or not. Probability theorists emphasize that there is no need for fuzzy mathematics, and that there isn't a single problem that probability cannot solve. Fuzzy theorists claim that there indeed is something called non-statistical uncertainty as distinct from probabilistic uncertainty. Over 500 research papers published have debated over this issue. However, we will leave polemics aside and assume that non-statistical uncertainty does exist.

that the colour of the sky is 'blue', then can we unambiguously define the concept 'blue'? It is difficult to do so given the multitude of shades of blue, and the fact that they form a continuum – fading smoothly from one shade to the next. Take another example. What do we mean when we say that a man is 'tall'? Is he 6 feet tall, or 7 feet? And if we agree that a 6 foot man is 'tall', does it imply that a person with a height 5 feet and 9 inches is 'not tall'? What about someone 6 feet one inch? There is obviously a problem with *classical bivalence* – the idea that things are either black or white – when we try to apply it to real world concepts in order to pin down a definition. There is an inherent *fuzzy uncertainty* in almost every concept we work with. As Kosko points out [1], there is a " ... mismatch problem: The world is gray but science is black and white."

We deal with this fuzzy uncertainty in the form of inexact information in the real world on a regular basis. Be it finding a place to park a car, deciding a route to a destination, or crossing a crowded street, we work with fuzzy uncertainty with ease. In many cases, imprecision seems to be better than precision. To quote an example from Bezdek [2]: "Suppose, as you approach a red light, you must advise a driving student to apply the brakes. Would you say 'begin braking 74 feet from the crosswalk'? Or would your advice be more like 'Apply the brakes pretty soon'?" I think the answer is clear.

Fuzzy uncertainty stems from lexical imprecision – an elasticity in the meaning of a concept. Real world concepts transition smoothly into one another rather than abruptly. Words are flexible in their meaning. And real world information is almost always partial. Human beings are able to deal with this uncertainty because of their ability to put together uncertain facts and *experience* into a reasoning framework that is approximate in nature. Our inexact calculus works with imprecise concepts. Humans reason with a *fuzzy logic*. To motivate ourselves further towards the need for modelling fuzzy uncertainty consider the following example. Is a forty year old middle-aged? Many of us strongly believe so. Is a fifty year old middle-aged? Suddenly we are unsure. To some extent, we say, a fifty year old is both middle-aged, as well as old. Things go fuzzy. From various members of a class or category, we seem to mentally extract a prototype that becomes an example that best represents that class for us. And when asked whether a person aged x years is middle-aged what we are really doing is trying to assess the extent to which x years is *compatible* with our mental prototype of the concept 'middle-aged'. Another way to look at the same thing is to ask, 'Given that you are middle-aged, what is the *possibility* that you are 45 years old'? Let's look at this belief, compatibility or possibility more closely.

Assume we have a concept MIDDLE-AGED which we wish to model, mathematically. Every one would almost certainly agree that a 40 year old is MIDDLE-AGED. In other words, to say that a person is MIDDLE-AGED is another way of saying that he is around 40 years old. So let us say that we agree that a 40 year old is MIDDLE-AGED to degree 100%. What about a 30 year old? Or a 45 year old? Our degree of belief that a 30 or a 50 year old is MIDDLE-AGED is lower than that of a 40 year old being MIDDLE-AGED. Here, age is a *linguistic variable*, taking on *linguistic values* such as YOUNG, MIDDLE-AGED, or OLD. One might quantify such linguistic values using the idea of a possibility. The possibility that a person is 30 years old, given that he is MIDDLE AGED, is 0.75. *Table 1* describes a set of possibilities of a person being x years of age given that he or she is MIDDLE-AGED.

Table 1. The possibility or belief that persons aged x years (from 0 to 80 years) are middle-aged. Our belief that a middle-aged person is 40 years old is 100%. Possibilities of other ages being compatible with 'middle-aged' diminish smoothly from 1 as we move away from 40 towards 80 or 0.

Age	Possibility/Belief /Compatibility
0	0.00
10	0.25
20	0.50
30	0.75
40	1.00
50	0.75
60	0.50
70	0.25
80	0.0

Table 1 summarizes a mapping from a *universe of discourse*, X, of ages, into degrees of belief that a person aged x years is MIDDLE-AGED. Mathematically, we have the *membership* of x in the *fuzzy set* MIDDLE-AGED as

$$\mu_{\text{MIDDLE-AGED}}(x): X \rightarrow [0,1].$$

The data of *Table 1* can be plotted as shown in *Figure 1*, and is called the *membership function* of the fuzzy set MIDDLE-AGED.

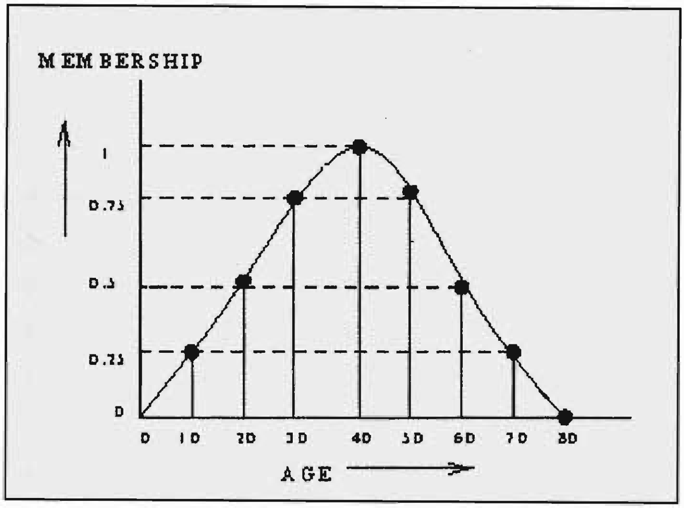

Figure 1. Plot of the beliefs or compatibilities of Table 1. The curve plotted signifies the membership function of a fuzzy set MIDDLE-AGED. Notice how the beliefs decrease smoothly away from 1 towards 0.

Fuzzy Sets and their Geometry

In a seminal 1965 paper entitled 'Fuzzy Sets' [3], Lofti A Zadeh, at the University of California, Berkeley, developed a mathematical framework that laid the foundations for today's fuzzy systems. Such systems have made their way into innumerable applications from the Sendai Subway control system, to washing machines and camcorders. As we discuss in Part 2, *fuzzy systems* are fast becoming ubiquitous and are here to stay.

The history of fuzzy sets can actually be traced back to Max Black, who published a paper on 'Vagueness: An Exercise in Logical Analysis" in 1937, in the journal *Philosophy of Science*. Black's paper defined the first simple fuzzy set. Even the famous philosopher Bertrand Russell pondered 'Vagueness' in a paper published in the *Australian Journal of Philosophy* in 1923. But it is Lofti Zadeh, who is to be credited with formally defining and building the field of fuzzy set theory, starting with his 1965 paper.

Formally speaking, a *fuzzy set* is defined on a universe of discourse (UOD) and maps the elements of the UOD into a set of real numbers which denote the membership of UOD elements in the set. In other words it is a function mapping UOD elements to membership values. This is similar to the definition of a characteristic function employed for classical sets, but with a difference. Classical sets admit memberships of either 0 or 1. Fuzzy sets admit a continuum of memberships *between* 0 and 1. We therefore have,

$$m(x): X \to \{0, 1\} \text{ for classical sets,}$$
$$\mu(x): X \to [0, 1] \text{ for fuzzy sets,}$$

where X is the universe of discourse. As suggested by Zadeh, the fuzzy set of *Figure 1* can be represented using the following notation:

$$M = 0/0 + 10/0.25 + 20/0.5 + 30/0.75 + 40/1.0 + 50/0.75 + 60/0.5 + 70/0.25 + 80/0$$

where the element value is separated from its membership by a '/', and the '+' indicates a union of elements that comprise the set. Here it is assumed that the universe of a discourse is a discrete one comprising the ages $X = \{0, 10, 20, 30, 40, 50, 60, 70, 80\}$. A word about notation. As an alternative, we may represent the element memberships more compactly as an ordered set:

$$M = \{0, 0.25, 0.5, 0.75, 1, 0.75, 0.5, 0.25, 0\}.$$

Here, it is assumed that the correspondence between the set of ages X and the set of membership values, M, is clear. The notation { } is used to denote a set of elements as well as the set of their memberships that are *assumed to be ordered in accordance with the sequence of elements in the set.* We use this notation freely, and the meaning of the contents of the set will be clear from the context.

The principal idea is to attempt to quantify the membership of an element of the UOD in the linguistically labelled set. So we may interpret the above representation to mean things like: a 40 year old is a member of the set MIDDLE AGED to degree 100%, whereas a 10 year old is a member only to degree 25%. In fuzzy set theory, *everything is a matter of degree*. In classical sets elements are either members or are not members of the set in question. In fuzzy sets, the elements are members of the set to certain degrees, inclusive of 0 and 1 which are the classical case of non-membership or full membership respectively.

As an example, consider two sets, one classical, the other fuzzy. Our classical set is: The set of numbers between 3 and 5 inclusive, or $N_c = \{x \in \mathbf{R} \mid 3 \leq x \leq 5\}$, where $m_{N_c}(x)\colon \mathbf{R} \to \{0, 1\}$. Here $\mathbf{R}$ denotes the set of real numbers. This set is plotted in *Figure 2*.

Our fuzzy set is: The set of numbers *close to 4*. Here $\mu_{N_f}(x)\colon \mathbf{R} \to \{0, 1\}$, and this set is plotted in *Figure 3*. Clearly, fuzzy sets

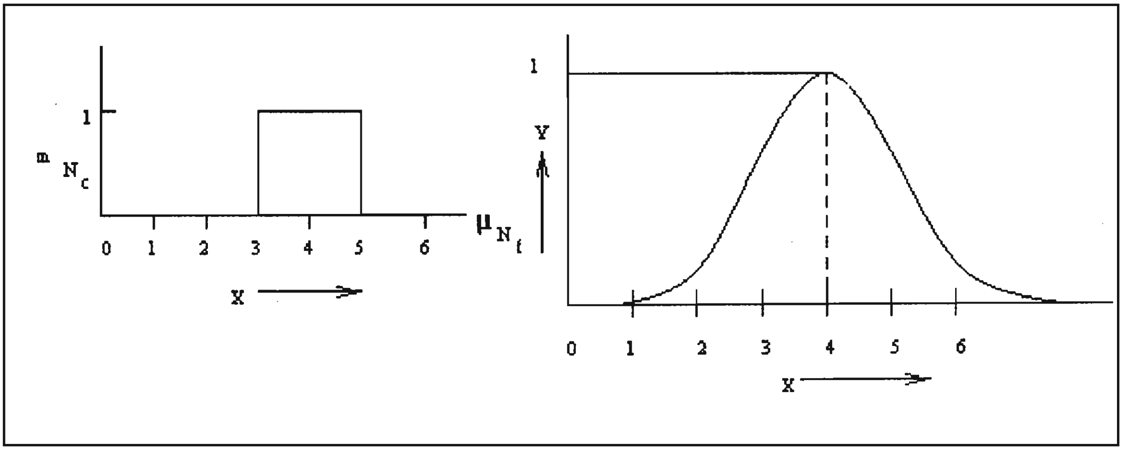

Figure 2. The membership function of a classical or crisp set marks an abrupt transition from 0 to 1 to include numbers in the interval [3,5] as members of the set and all other numbers as non-members.

Figure 3(right). The fuzzy set of numbers close to 4. Numbers further away from 4 have lesser membership that numbers closer to 4. Memberships vary smoothly from 1 towards 0 as we move away from 4.

admit a continuum of memberships whereas classical sets admit only 2-state membership.

As shown in *Figure 4*, a fuzzy set can be given an interesting geometrical interpretation. Consider a universe of two objects, A and B. In this geometry, we assume that each axis describes the degrees of membership of objects A and B respectively in a set (or a fuzzy set), P.

Coordinate (0, 0) is interpreted as $A/0 + B/0$, implying that neither A nor B are members of P, i.e., $P=\phi$ or the null set.

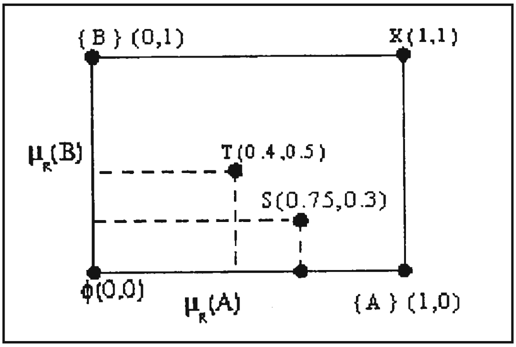

Figure 4. The geometry of fuzzy sets. Corners of the unit square represent the classical power set 2^X. Fuzzy sets are the points that lie within the square.

Coordinate (1, 1) is interpreted as $A/1 + B/1$, where both A and B are members of R, i.e., $R = \{A, B\} = X$. Therefore, coordinates (1, 0) and (0, 1) are interpreted as representing singleton sets $\{A\}$ and $\{B\}$ respectively. It is important to understand that the classical power set $2^X = \{\{\}, \{A\}, \{B\}, \{A,B\}\}$ is represented by the four corners of the unit square (commonly denoted as $\mathbf{B}^2$).

In contrast, the point S in *Figure 4* with coordinates (0.75, 0.3) represents a fuzzy set $S = A/0.75 + B/0.3$ with the connotation: A and B are members of S to degrees 0.75 and 0.3 respectively. In the present example, the points that lie within the square (denoted $I^2 = [0,1] \times [0,1]$) represent fuzzy sets. The set of all such possible (infinite) points in I^2 comprise the fuzzy power set F^X. Note that $\mathbf{B}^2 \subset \mathbf{I}^2$. Classical sets are thus special cases of fuzzy sets.

The geometric interpretation of a fuzzy set is an outcome of Bart Kosko's PhD dissertation "Foundations of Fuzzy Estimation Theory", University of California, Irvine, 1987. A very accessible treatment of the subject (as well as other related topics) can be found in Kosko's book: *Fuzzy Thinking* (Hyperion, 1993).

Simple Operations on Fuzzy Sets

The operations of intersection ($\cap$), union ($\cup$), and complementation (′) that we are so familiar with in classical set theory carry over to fuzzy set theory in a rather direct fashion. Consider the fuzzy sets S and T in *Figure 4*:

$$S = \{0.75, 0.3\} = A/0.75 + B/0.3$$
$$T = \{0.4, 0.6\} = A/0.4 + B/0.6$$

The intersection operator should indicate what is common to both sets. If A is member of S to degree 0.75 and A is a member of T to degree 0.4, then A is a member of both sets S and T at least to degree 0.4. Intersection thus translates to 'minimum' (or min). As a result, we have $\mu_{S \cap T}(A) = \min(\mu_S(A), \mu_T(A)) = \min(0.75, 0.4) = 0.4$ and $\mu_{S \cap T}(B) = \min(\mu_S(B), \mu_T(B)) = \min(0.3, 0.6) = 0.3$ or $S \cap T = \{0.4, 0.3\} = A/0.4 + B/0.3$.

Interestingly, the same logic works for classical sets. Consider the two classical sets $X = \{A, B\}$ and $Y = \{A\}$. Clearly, $\mu_X(A) = \mu_Y(A) = 1$; $\mu_X(B) = 1$ and $\mu_Y(B) = 0$. Then $\mu_{X \cap Y}(A) = \min(1,1) = 1$; $\mu_{X \cap Y}(B) = \min(1, 0) = 0$. Therefore $X \cap Y = \{1, 0\} = A/1 + B/0$ which represents the singleton set $\{A\}$.

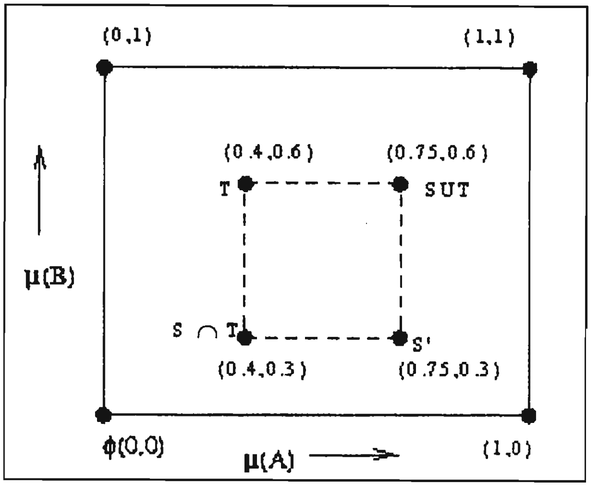

Figure 5. Fuzzy intersections and unions portrayed geometrically.

Similarly the union operator tells us the extent to which elements may at most be members of a set. Here, union translates to the 'maximum' (or max) operator. Continuing with the same example, $\mu_{S \cup T}(A) = \max(\mu_S(A), \mu_T(A)) = \max(0.75, 0.4) = 0.75$, and $\mu_{S \cup T}(B) = \max(\mu_S(B), \mu T(B)) = \max(0.3, 0.6) = 0.6$, or $S \cup T = \{0.75, 0.6\} = A/0.75 + B/0.6$.

Once again we may employ the same logic as for classical sets to obtain their union. If $Y = \{A\}$ and $Z = \{B\}$ over a universe $X = \{A, B\}$, we may rewrite these sets in membership notation as: $Y = \{1, 0\} = A/1 + B/0$; $Z = \{0, 1\} = A/0 + B/1$. Then $Y \cup Z = \{\max(\mu_Y(A), \mu_Z(A)), \max(\mu_Y(B), \mu_Z(B))\} = \{1, 1\} = A/1 + B/1 = X$. And indeed this is the expected result.

Resorting to a geometric interpretation again, *Figure* 5 shows an interesting picture that emerges. The four sets $S, T, S \cup T, S \cap T$ form a rectangle with the union at the upper right corner and the intersection at the lower left. *Figure* 6 shows the two operations of intersection and union graphically on two triangular shaped fuzzy sets. This diagram helps develop an intuitive feel for these two operations.

We return to geometry in a moment, but first we introduce the third fundamental operation (after set intersection and union) – set complementation. Consider a fuzzy set, $S = \{0.75, 0.3\}$. Its complement must describe what is outside the set – between the

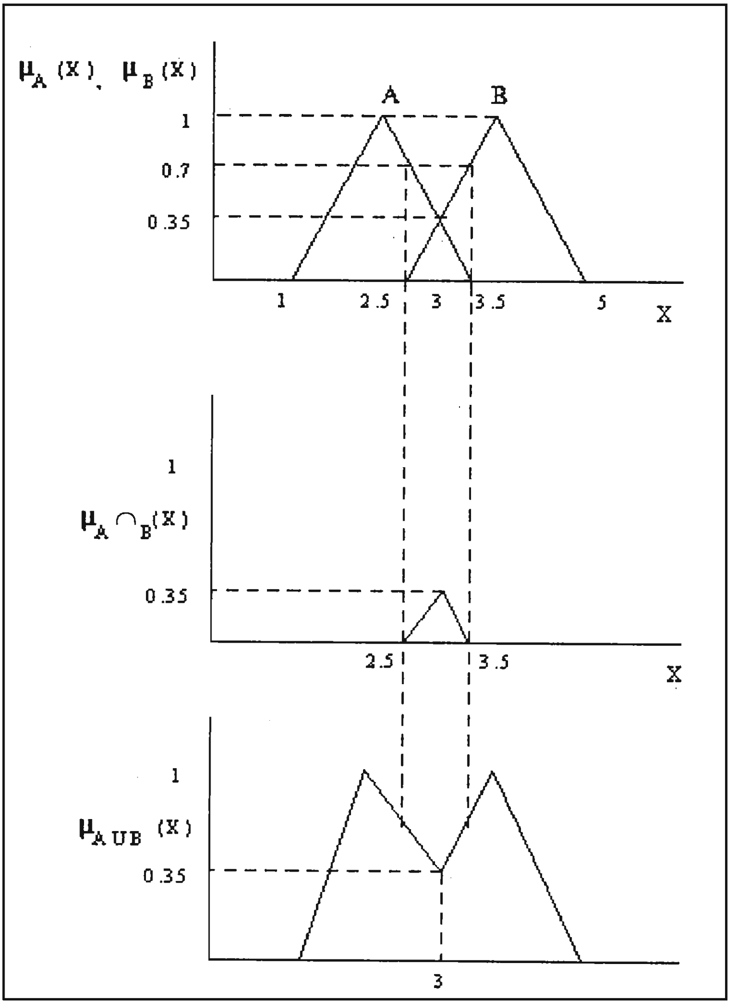

Figure 6. Fuzzy intersections and unions portrayed graphically.

set and its universe. In terms of memberships, this translates to saying $\mu_{S'}(A) = 1 - \mu_S(A) = 1 - 0.75 = 0.25$, and $\mu_{S'}(B) = 1 - \mu_S(B) = 1 - 0.3 = 0.7$. Therefore $S' = \{0.25, 0.7\} = A/0.25 + B/0.7$.

We perform a similar operation each time we take the complement of a classical set. Consider $Y = \{A\}$ on a universe $X = \{A, B\}$. Then $Y = \{1, 0\} = A/1 + B/0$; and $Y' = \{(1-1), (1-0)\} = \{0, 1\} = A/0 + B/1$. The complement of $\{A\}$ is $\{B\}$. *Figure* 7 portrays the complementation operator graphically on a triangular fuzzy set.

Now back to geometry. *Figure 8* depicts the four sets S, S', $S \cap S'$, $S \cup S'$. Once again we see the familiar membership

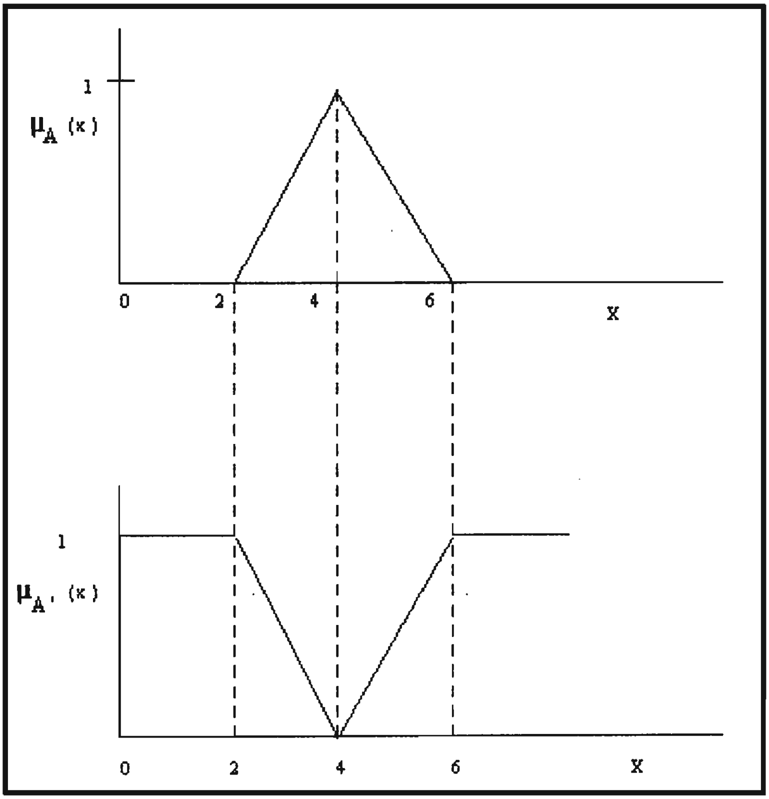

Figure 7. Set complementation portrayed graphically.

rectangle, but now with a difference. There is a symmetry about the midpoint of the cube. In this geometry, notice something even more strange: $S \cup S' \neq X$ and $S \cap S' \neq \phi$! This completely contradicts what we learnt at school. In fact these two laws – the Law of Non-contradiction and the Law of Excluded Middle – which have been around from the times of Aristotle, are always *violated* for fuzzy sets. They hold only for classical sets.

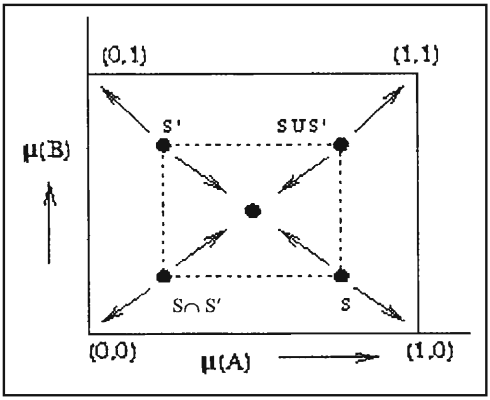

Figure 8. The midpoint of the unit cube is maximally fuzzy. The corners are non-fuzzy.

Property/Operation	Definition
support (non-zero membership points in A)	$\text{supp} A = \{x \in X \mid \mu_A(x) > 0\}$
height (the maximum membership in A)	$\text{hgt} A = \max\{\mu_A(x)\}$
α – cut (Points with membership greater than α)	$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$
cardinality (sum of element memberships in A)	$\lvert A \rvert = \Sigma_{x \in X} \mu_A(x)$
subset	$A \subset B$ iff $\mu_A(x) \leq \mu_B(x)$ for every $x \in X$

Table 2.

Geometry provides us a graceful explanation. As S becomes increasingly non-fuzzy, i.e., as it moves, say, towards the coordinate (1,0) as indicated by the arrow in *Figure 8*, sets S', and $S \cup S'$ respectively move toward the corners (0,1), (0,0) and (1,1). What we are really saying is that S has an increasingly dominant membership of A and a less dominant membership of B. $S \cap S'$ therefore becomes increasingly similar to the null set where as $S \cup S'$ increasingly looks like the classical universe X.

In contrast, as S becomes increasingly fuzzy, its component memberships tend to 0.5 (and we are increasingly unsure whether objects A and B are indeed members of S or not). Simultaneously, S', $S \cap S'$, $S \cup S'$ also move towards (0.5,0.5). At the midpoint, $S = S' = S \cap S' = S \cup S' = A/0.5 + B/0.5$. The midpoint of the cube is *maximally fuzzy*.

Table 2 summarizes important properties and operations that can be defined on fuzzy sets. These operations or properties are basically of mathematical interest. In practical applications the most common operations employed are those of intersection, union, and complementation.

Conclusions

In this part of the article we have identified fuzzy uncertainty as being distinct from probabilistic uncertainty. We have understood the notion of a fuzzy set, and seen important operations that can be performed on such sets. Lastly, a geometrical interpretation of the fuzzy set and its attendant operations were discussed. In Part 2, we will develop a framework for reasoning based on fuzzy sets, and review a simple application.

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