Uncertainty Principles and Fourier Analysis

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To quote the mathematician G B Folland (see [2]): "The uncertainty principle is partly a description of a characteristic feature of quantum mechanical systems, partly a statement about the limitations of one's ability to perform measurements on a system without disturbing it, and partly *a meta-theorem in harmonic analysis that can be summarized as follows: A nonzero function and its Fourier transform cannot both be sharply localized.*"

It is the last part of the paragraph that is the raison d'etre for the mathematician's interest in uncertainty principles. Another way to express the *meta uncertainty principle* is: A nonzero function and its Fourier transform cannot both be sharply *concentrated.* Depending on the definition of *concentration,* one gets various *avatars* of the meta uncertainty principle. Due to limitations of space, we present here only three such, and without too many proofs!

In what follows, we assume a knowledge of basic Fourier analysis on the part of the reader. Those who are not familiar with Fourier analysis are encouraged to look up *Box 1* along with [3].

(A) *Heisenberg's inequality*: Let us measure concentration in terms of *standard deviation* i.e. for a square integrable function defined on $\mathbb{R}$ and normalized so that $\int_{-\infty}^{\infty} |f(x)|^2 dx = 1$, and any $a \in \mathbb{R}$, consider the quantity $\int_{-\infty}^{\infty} (x-a)^2 |f(x)|^2 dx$. (To convince herself that the more concentrated f is around a, the smaller this quantity will be, the reader is encouraged to solve the following easy exercise: Suppose $\int_{-\infty}^{\infty} |f(x)|^2 dx = 1$ and f is zero outside the interval $[a-l, a+l]$. Prove that if $l \to 0$, then the quantity $\int_{-\infty}^{\infty} (x-a)^2 |f(x)|^2 dx \to 0$). Let $\hat{f}$

be the Fourier transform of f, i.e. $\hat{f}(y) = \int_{-\infty}^{\infty} e^{-2\pi i y x} f(x)dx$. (*Warning:* Note the slightly non-standard definition of the Fourier transform!) In view of the *Plancherel theorem* (see e) of *Box 1*), we also have $\int_{-\infty}^{\infty} |\hat{f}(y)|^2 dy = 1$. Then no matter which point $b \in I\!R$ we choose, $\hat{f}$ cannot be concentrated around b, if f is concentrated around a. More precisely,

$$\left(\int_{-\infty}^{\infty} (x-a)^2 |f(x)|^2 dx\right)\left(\int_{-\infty}^{\infty} (y-b)^2 |\hat{f}(y)|^2 dy\right) \geq \frac{1}{16\pi^2} \qquad (*)$$

(Exercise: What does the inequality become if we do *not* assume $\int_{-\infty}^{\infty} |f(x)|^2 dx = 1$?)

Box 1. Basic Facts about the Fourier Transform

For 'reasonable' functions, we list the following useful facts about the Fourier transform:

a) If a, b are scalars, and f, g functions, $(af + bg)^{\wedge} = a\hat{f} + b\hat{g}$

b) If $g(x) = f(x + x_0)$, then $\hat{g}(y) = e^{2\pi i y x_0} \hat{f}(y)$

c) If $h(x) = e^{2\pi i x_0 x} f(x)$, then $\hat{h}(y) = \hat{f}(y - x_0)$

d) $(f')^{\wedge}(y) = (2\pi i y)\hat{f}(y)$, where f' is the derivative of f.

e) For any square integrable g, $\int_{-\infty}^{\infty} |g(y)|^2 dy = \int_{-\infty}^{\infty} |\hat{g}(y)|^2 dy$. Hence, using d),

$$\int_{-\infty}^{\infty} |f'(y)|^2 dy = \int_{-\infty}^{\infty} 4\pi^2 y^2 |\hat{f}(y)|^2 dy.$$

f) Fourier inversion formula : If f is 'sufficiently nice' (for example, f continuous and integrable and $\hat{f}$ integrable), then

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(y) e^{2\pi i y x} dy$$

The above shows that if we have a sequence of f's which are concentrated more and more around a, i.e. the first quantity in the inequality goes to zero, then the second quantity for these f's in the left hand side of the inequality must necessarily blow up, no matter what b is. An interesting question that comes up is: When is equality attained in (∗)? The surprising answer is that equality is attained if and only if f is, modulo translation and phase change, a Gaussian (i.e. a function of the form Ae^{-cx^2}). The student-reader is encouraged to ask her physics teacher why (∗) is essentially the celebrated Heisenberg uncertainty principle in disguise! See *Box 2* for a sketch of the proof of (∗).

(B) *Benedicks's theorem* : If we think of concentration in terms of f 'living' entirely on a set of finite measure, then we have the following beautiful result of M Benedicks: Let f be a *nonzero* square integrable function on $\mathbb{R}$. Then the Lebesgue measures of the sets $\{x : f(x) \neq 0\}$ and $\{y : \hat{f}(y) \neq 0\}$ *cannot* both be finite. (For those who are not familiar with the jargon of measure theory, a (measurable) subset $A \subseteq \mathbb{R}$ is of finite measure, if it can be covered by a countable union of intervals I_k such that $\sum_k$ (length of I_k) $< \infty$.) The result above is a significant generalization of the fact, well known to communication engineers, that a nonzero signal cannot be both *time limited and band limited.*

(C) *Hardy's Uncertainty Principle* : The rate at which a function decays at infinity can also be considered a measure of concentration. The following elegant result of Hardy's states that both f and $\hat{f}$ cannot be 'very rapidly' decreasing: Suppose f is a measurable function on $\mathbb{R}$ such that $|f(x)| \leq Ae^{-\alpha\pi x^2}$ and $|\hat{f}(y)| \leq Be^{-\beta\pi y^2}$ for some positive constants A, B, α, β. Then, if $\alpha\beta > 1$, f must necessarily be the zero function. (If $\alpha\beta = 1$, then the only functions satisfying the above inequalities are functions of the form $Ae^{-\alpha\pi x^2}$ Once again the ubiquitous Gaussian enters the picture!)

While all three theorems mentioned above reflect the same 'philosophy', it must be emphasized that each has to be *proved* separately. For an extensive bibliography of uncer-

Suggested Reading

[1] H Dym and H P McKean, *Fourier Series and Integrals.* Academic Press. New York, 1972.

[2] G B Folland and A Sitaram. The Uncertainty Principle: A Mathematical Survey. *The Journal of Fourier Analysis and Applications.* Vol. 3, No. 3. pp. 207–238, 1997.

[3] A Sitaram and S Thangavelu, From Fourier series to Fourier transforms. *Resonance.* Vol.3.pp.3–5, October 1998.

Box 2

We give here a brief sketch of the proof of $(*)$.

In what follows, let us assume that both f and $\hat{f}$ vanish at ∞ sufficiently rapidly and are smooth enough for all our calculations to make sense. Assume, for simplicity, that f is real valued, although we can easily dispense with this assumption. By translation and multiplication by a phase factor (see *Box 1*), we can assume that $a = 0$ and $b = 0$. So it is enough to show that if $\int_{-\infty}^{\infty} |f(x)|^2 dx = 1$, then

$$\left(\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx\right)\left(\int_{-\infty}^{\infty} y^2 |\hat{f}(y)|^2 dy\right) \geq \frac{1}{16\pi^2} \cdots\cdots \quad (**)$$

To prove $(**)$, consider $-\int_{-\infty}^{\infty} x f(x) f'(x) dx$. By integration by parts and using the fact that f vanishes at ∞ sufficiently rapidly, the above expression is just $\frac{1}{2}\int_{-\infty}^{\infty} (f(x))^2 dx$, and since $\int_{-\infty}^{\infty} (f(x))^2 dx = 1$, we have : $\frac{1}{2} =$

$$\left|\int_{-\infty}^{\infty} x f(x) f'(x) dx\right| \leq \int_{-\infty}^{\infty} |x f(x)||f'(x)| dx \leq \left(\int_{-\infty}^{\infty} x^2 (f(x))^2 dx\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |f'(x)|^2 dx\right)^{\frac{1}{2}}$$

The last inequality follows from the Cauchy–Schwarz inequality. Using d) and e) of *Box 1*, the last expression is just $\left(\int_{-\infty}^{\infty} x^2 (f(x))^2 dx\right)^{1/2} \left(4\pi^2 \int_{-\infty}^{\infty} y^2 |\hat{f}(y)|^2 dy\right)^{1/2}$, and the proofs of $(**)$ and $(*)$ follow. We should add that the proof of $(*)$ without the rather restrictive assumptions on f and $\hat{f}$ is not entirely trivial, and the reader is encouraged to look up [1] for a complete proof. One can also give a slick 'operator theoretic' proof, but in the interest of keeping the exposition elementary we have refrained from presenting it here.

tainty principles in mathematics, the reader may consult [2]. Finally, we should add that the *meta-uncertainty principle* is a meta-theorem, not only in Fourier analysis on $\mathbb{R}$ or $\mathbb{R}^n$, but also holds in harmonic analysis on much more general spaces (see [2]).

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