# Crossing Bridges

*Joseph Samuel*

Here is a figure (*Figure 1*) that many of you may have seen before. The problem is to trace the figure without removing your pen from the paper or retracing a path.

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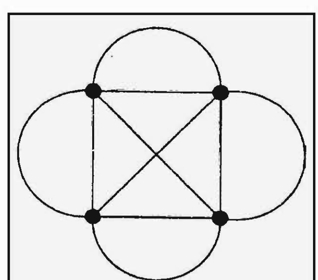
You have probably wasted hours during your school days trying to figure out this puzzle. I know I did. I have never been able to solve this puzzle. Someone I know did claim that he had done it, but on closer inspection it turned out that he had used sleight of hand. I don't know anyone who has succeeded. It seems fair, perhaps, to conclude that it cannot be done. Can we let the matter rest at that?

No! We cannot. The problem in a word, is Hungarians. The Hungarians are a small and extremely talented community of people (see *Box 1*). It is entirely possible that tomorrow a clever Hungarian will walk through the door and say 'See, this is how it is done'. Can we *show* that it cannot be done? What we are looking for here is a *proof*.

## The Seven Bridges of Königsberg

Königsberg is a town through which flows the river Pregel. There are seven bridges across the river as shown in the map (*Figure 2*). There was a saying among the inhabitants of this town that it was impossible to plan a walk so that each bridge was traversed once, but not more than once. Opinion was divided on whether the saying was correct or not. Some held that it was. Others were doubtful; but there was no one who maintained that is was actually possible. We are up against Hungarians again! Hungary is not too far from Königsberg and it is entirely possible that some visiting Hungarian will, on hearing about the challenge, spin on his heel, mutter something in Hungarian and traverse each bridge exactly once, following a path that no one had thought of up till then. Again, we need to

***Figure 1. School child's puzzle.***



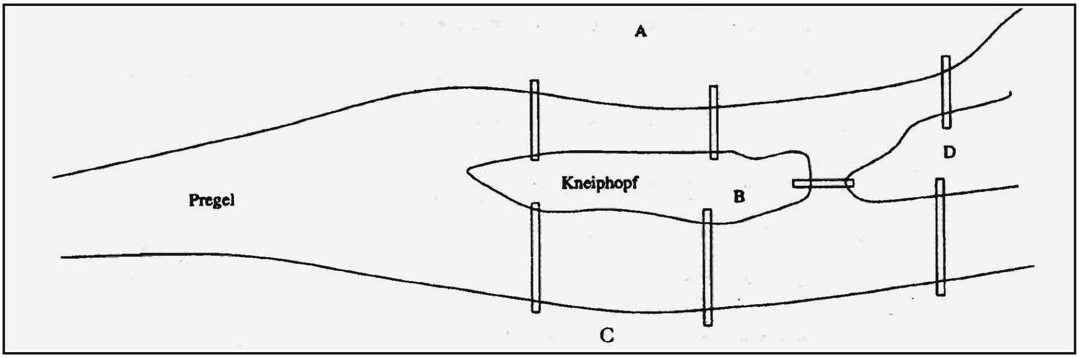



*Figure 2. Map of Königsberg.*

find a way to *prove* that it cannot be done. Till we do this we are not safe from Hungarians. The mere fact that we have so far been unable to do it means nothing. Perhaps we haven't tried hard enough. Perhaps we are not smart enough.

**Box 1. Hungarians.**

The parable of the Hungarians in the text is only meant to be a joke. No offence is intended towards any group of people – Hungarians, Poles, Cossacks, Chinese, Indians, or anyone. Hungarians are used here as they have developed a reputation for cleverness. As an example, note that the "Hungarian quartet", Leo Szilard, Eugene Wigner, Jancsi von Neumann and Edward Teller were all born within a decade in Budapest. But as Alexander Korda once remarked, "It's not enough to be Hungarian, you have to be talented too".

How would we go about establishing that the problem has no solution? One possibility is to enumerate all possible paths and show that none of them succeed. But this is very tedious and impractical. Even if we do this for the bridges of Königsberg, we would learn nothing about other similar situations. Consider for example the fictitious city of Lutetia at the confluence of rivers with two islands in the river called Geometria and Topologia (*Figure 3*).

Having solved the Königsberg problem by enumeration does not at all help us in understanding the more complex situation prevailing in Lutetia. We need to do better than that! We need to solve the entire class of problems at one shot. It is interesting to note that the map of Königsberg has a lot of redundant information. The map tells us the size of the island. This is clearly not relevant to the problem. Let us shrink the island down to a dot. Similarly, all land masses can be shrunk down to dots. The lengths of the bridges are also not relevant, just as the kind of vegetation growing on the island is not relevant. The widths of the bridges are also not relevant. Let us shrink the bridges down to lines. After getting rid of excess information, the map of Königsberg is reduced to a skeleton (*Figure 4*). It consists of four dots representing the land masses ABCD and

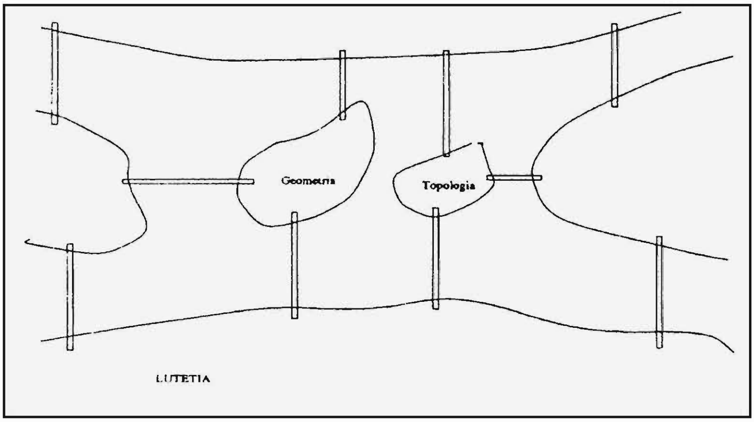



*Figure 3. Map of Lutetia.*

seven lines representing the bridges connecting them. This map has exactly the information we want. The problem now is 'Can I trace out the figure above without retracing a line?' The problem of the Königsberg bridges is clearly in the same class as the problem you 'wasted' your time on at school.

You may be heartened to learn that the problem of the Königsberg bridges was first solved by the great mathematician Leonhard Euler and this led to the birth of a new branch of mathematics – topology. We have now posed three problems in topology. What distinguishes these problems from other mathematical problems?

In school you are often asked to solve problems where both the data given and the answer sought are numbers: Eg. A B C and D dig a ditch. After a brief description of the abilities of A B C and D in digging ditches, one is asked how long the job will take to

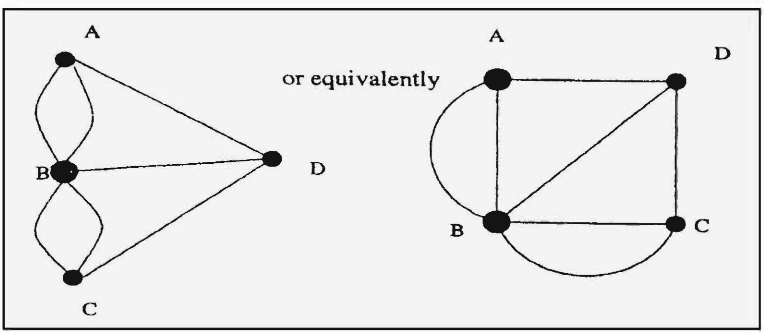



*Figure 4. Skeleton of the map of Königsberg.*

get done. (When I was at school, A was in his prime; indeed, according to one eminent authority, A could do as much work in an hour as B in two or C in five!) The answer that you have to find is a time which is, say five hours. In other problems one may compute a distance or the sum of money in a bank account after various withdrawals, deposits and interest rates are accounted for.

The problems posed above are in a different class. The data given are not numbers, but a set of points and connections between them. The answer sought is not a magnitude. The question is : Can you trace these lines? The answer is yes or no. Both the questions and the answers are of a qualitative nature, not of a quantitative nature.

Problems like the Königsberg bridge problem are studied in topology. More precisely, network topology.

## Network Topology

A **network** is a set of points (called **vertices**) with interconnections. The interconnections are called **edges.** If one can go from any vertex of the network to any other vertex by following edges, we say that the network is connected. We always suppose that the number of vertices and edges is finite. *Figures* (*1, 4*) are

**Box 2. Geometry and Topology.**

Let us take a moment to understand the difference between topological and geometrical properties. If you make a regular polyhedron out of wire and sit on it so that it is squashed (but not broken!) the result is a topologically regular polyhedron. The faces are no longer regular polygons and therefore the polyhedron is not geometrically regular. But topologically, all we care about is that the squashed polyhedron has the same number of edges meeting at every vertex and the same number of edges around every face. It is interesting that in order to solve problems you sometimes 'throw away' data. In classifying the regular solids, we did not use all the information at our disposal. Though we were interested in polyhedra which were geometrically regular, we only used the fact that these polyhedra were topologically regular. None of the geometrical properties of regular solids were used. They were not necessary. Topological information was sufficient to provide a solution. In the problem of the Königsberg bridges it is quite obvious what the irrelevant data is. This is not always so. In cosmology for instance, progress was hampered for centuries because we did not realise that we ourselves were irrelevant!

examples of connected, finite networks. The question we posed is now easily tackled.

We wish to prove that there is no path in the network (1) which traverses the edges of the network exactly once. We prove this by assuming that there *is* such a path and arriving at a contradiction. Let us suppose that there is a path in a network that traverses each edge exactly once. At all vertices except the starting and ending vertices, the path will arrive by an edge and leave by *another* edge. Thus the total number of edges at any vertex must be even unless it is the starting or the ending vertex. We conclude that if the path is closed (the starting and ending vertices are the same) all vertices must have an even number of edges. If the path is open, two vertices must have an odd number of edges and the rest must have an even number. The network must have either exactly two vertices with an odd number of edges or no such vertices. The network in *Figure 1* has four vertices with an odd number (five) of edges and it follows that the problem posed at the beginning of this article does not have a solution. The reader can now prove that the bridges of Königsberg and Lutetia are both safe from Hungarians.

## Planar Networks

Planar networks are networks that you can draw on a plane without the edges crossing. An example of a planar network is *Figure 4*. *Figure 1* looks non-planar as drawn but you can redraw it so that the edges do not cross. So it is, in fact, planar. Planar networks divide the plane into regions. Each of the regions we call a **face**. We write $V$ for the number of vertices, $E$ for the number of edges and $F$ for the number of faces. Consider a finite, connected and planar network. We now reduce it by the following elementary moves. These moves reduce the number of edges while keeping the network finite, connected and planar.

The first move (see *Figures 5*) reduces $E$ and $F$ by one. The second (see *Figure 6*) reduces $V$ and $E$ by one. Neither of the

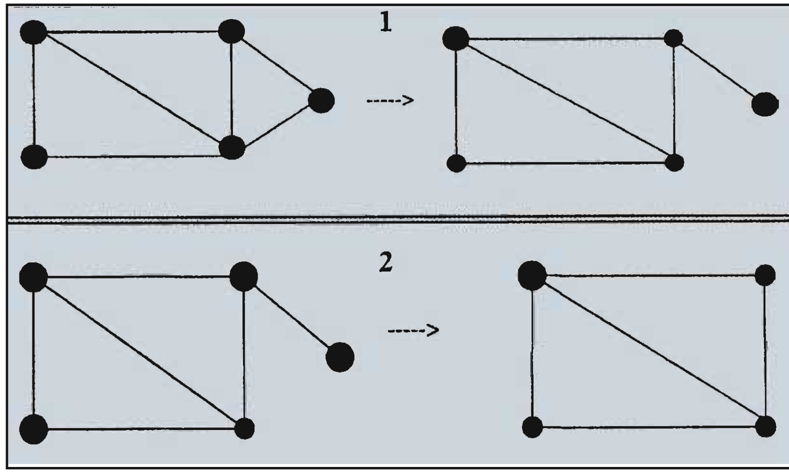



*Figure 5. Remove an outside edge and a face along with it.*

*Figure 6. Remove a vertex that is dangling from an edge along with that edge.*

moves alters the quantity $V-E+F$. A network can be reduced by the above moves until there are no more edges. The original network is now reduced to a single vertex, i.e. a network with $V=1, E=F=0$. We conclude that for all planar networks, $V-E+F=1$.

You might have thought of the school child's puzzle as a means of whiling away time in class when serious matters like history, geography and calculus were being taught. In fact, as we have seen, great minds like Euler's have also been absorbed in similar questions. The reason (apart from the natural tendency of active minds to 'play' with ideas) is that these apparently simple (topological) ideas are actually very deep and have applications in diverse fields like physics and geometry. You will see below how the study of network topology helps us to understand a question in geometry: the classification of the regular solids.

## Platonic Solids

Let us move on now from Königsberg and topology to Greece and geometry. The ancient Greeks were geometers *par excellence*. They discovered the five regular solids (see *Box 3*), also called the platonic solids. On the plane one can draw regular polygons – polygons with identical sides and angles. In three dimensions, we talk of polyhedra rather than polygons. Imagine taking a watermelon and slicing away the skin with a finite number of plane cuts. Each cut takes

**Box 3. Definitions and Such.**

Regular polyhedra are variously defined by various authors. Our use of the term agrees with Coxeter (see Suggested Reading below). On page 5, Coxeter defines a convex polyhedron to be regular if "its faces are regular and equal and its vertices are all surrounded alike". The alert reader would have noticed that in the main article we went to extreme lengths to avoid defining anything precisely. In fact we defined a polyhedron in terms of a watermelon without ever defining a watermelon! Readers dissatisfied with this slipshod approach are advised to consult their friendly neighbourhood mathematician or the excellent book by Coxeter. Without saying so explicitly, we have assumed here that the polyhedra of interest are convex (I very much doubt that all watermelons are convex) and that their surfaces are simply connected (I believe this is true for all watermelons).

some of the flesh along with the skin and exposes a 'face'. The resulting figure is called a *polyhedron* (many faces, the word 'hedra' means 'face' in Greek). This is quite similar to producing polygons by taking a disk and cutting away the circumference by a finite number of straight lines. The faces of a polyhedron meet at lines called edges. The edges meet at points called vertices. We define a polyhedron to be regular if it has faces which are identical regular polygons and if it has the same number of edges meeting at every vertex. These regular polyhedra are called *platonic* solids. A solid cube is an example. Are there others? Yes, there is the tetrahedron. With some playing around you can produce three more – the octahedron, the dodecahedron and the icosahedron. Are there still more? We are unable to produce more. But that doesn't prove that there are no more regular solids. Remember the parable of the Hungarians? We need to prove that there are no more regular solids.

This is a problem in geometry. But as we will see, our study of topology and planar networks will help us to answer it. Take the surface of any polyhedron (not necessarily regular), remove one of the faces, and open it out and spread it on a plane. It is a connected, planar network. Since we had to remove a face to make it a planar network, it follows that for the original solid figure the number of vertices, edges and faces are then related by

$$V - E + F = 2 . \qquad (1)$$

This is a famous formula discovered by Euler (and independently by Descartes). How does this help us show that there are no more regular figures? Well, regular figures have the property that all faces and all vertices are identical. The number of edges meeting at each vertex must be the same (let us say $p$) for all vertices. Similarly, the number of edges surrounding a face (say $q$) must be the same for all faces. (Convince yourself that $p$ and $q$ must be greater than two.) It follows that

$$E=pV/2\,, \qquad (2)$$

since one can count the edges by multiplying the number of edges at each vertex by the number of vertices and dividing by two (since each edge connects two vertices and has been counted twice). Similarly

$$E = qF/2. \qquad (3)$$

Using Euler's formula (1) we find,

$$1/p + 1/q = 1/2 + 1/E. \qquad (4)$$

Remember that $p$ and $q$ are integers and must be three or greater. We now need to find all the solutions of (4). If both $p$ and $q$ are four or more, then the LHS is less than or equal to 1/2. There is no solution for any number of edges. Thus either $q$ or $p$ must be equal to 3. Let us suppose that $q = 3$. Then (4) reads

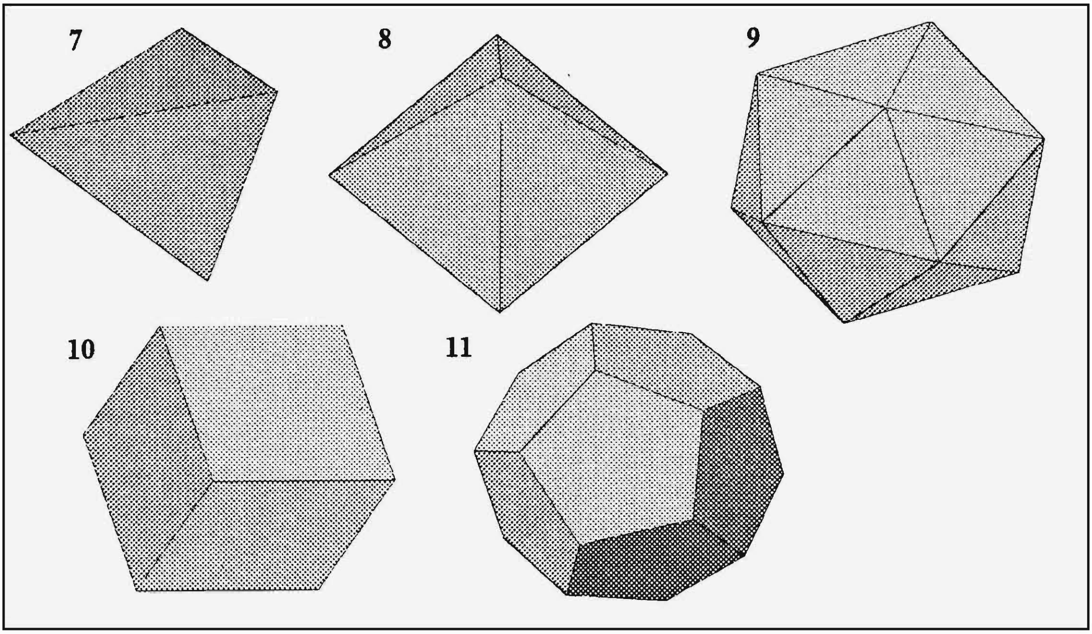
$$1/p = 1/6 + 1/E.$$

Clearly $p$ must be less than 6, otherwise there is no solution for $E$. The only possibilities are $p = 3, 4, 5$. Enumerating these possibilities,

$q=3$, $p=3$: from equations (2, 3, 4), $E=6$, $V=4$, $F=4$, this describes the tetrahedron [1] (*Figure 7*).
$q=3$, $p=4$: $E=12$, $V=6$, $F=8$, this is the octahedron[1] (*Figure 8*).
$q=3, p=5$: $E=30, V=12, F=20$ this is the icosahedron[1] (*Figure 9*).

[1] It still remains to be proved though that this is the only figure with these values of $V$, $E$ and $F$. This takes a bit more work and will be dealt with in a future issue of *Resonance*.

The other solutions, in which $p=3$, can be got by switching $p$




*Figures 7–11: Tetrahedron; Octahedron; Icosahedron; Cube; Dodecahedron.*

and $q$ in the above solutions. This give two more new ones

> $q=4, p=3$: from equations (2, 3, 4) $E=12$, $V=8$, $F=6$, this is the cube (or hexahedron[1]) (*Figure 10*).
> $q=5, p=3$: $E=30, V=20, F=12$ this is the dodecahedron[1] (*Figure 11*).

Thus the solutions to Euler's formula (which is a formula from topology) reproduce exactly the five known regular figures occurring in geometry (*Figures 7–11*). Regular figures with values of $V, E$ and $F$ other than those above do not exist! You see that one can use ideas from one field (topology) to address problems in another (geometry). Similarly, topology and geometry also have applications in physics. Our understanding of one field is often improved by crossing bridges to other fields!

## Suggested Reading


[1] James Newman. *The World of Mathematics*. Simon and Schuster . NY, 1956.
[2] Ian Stewart. *Concepts of Modern Mathematics*. Penguin. NY, 1976.
[3] H S M Coxeter. *Regular Polytopes*. Dover NY, 1973. (For more advanced readers.)



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