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The Indian Gold Rush

It was a great moment for Indians, when we got three gold and three silver medals in the 39^{th} *International Mathematical Olympiad*, held in Taipei, Taiwan in July '98. It was well organised in almost all aspects. First of all, the problems in the final paper were of good standard. We had nice excursions, good cultural programmes, good guide,... etc., though the food situation was rather difficult for us, especially the vegetarians amongst us (Rishi Raj, Yogananda and I survived on fruits).

Our performance was really great. We once again proved our strength in Geometry. India was the only country to get all six full 7 points in three problems. After the first day's examination was over in which all of us had got two problems right we were pleasantly surprised to hear our deputy leader, Yogananda, tell us that the second problem was a proposal from India! We, of course, did not know this. We are proud of R B Bapat, Indian Statistical Institute, Delhi, who was the author of this problem. More than 250 contestants got zero in that problem in which our team scored 42 out of 42! The one thing that is not in favour of us is that none of us could solve the 3^{rd} and the 6^{th} problem completely.

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Here , I briefly discuss the problems which appeared in **IMO**:

Problem 1. *In the convex quadrilateral $ABCD$, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. Suppose that the point P, where the perpendicular bisectors of AB and DC meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles ABP and CDP have equal areas.*

Our team found this problem rather easy and altogether gave five different solutions, two using co-ordinate geometry, one using calculus and the other two being pure geometry solutions. If the quadrilateral is cyclic, then it is easy to see that the triangles APB and CPD have same area. While proving the other way, I assumed that the result is not true and came to a contradiction. If the result is not true, then, assume $AP > CP$ (since they are not equal). We draw a circle with P as centre and PA as radius. Then extend the diagonals to intersect the circle again at X and Y Thus triangle XPY = triangle APB = triangle CPD (this follows from the previous implication), which leads to a contradiction.

Chetan and Soham solved by proving the fact that $OMPN$ is a parallelogram, where O is the point of intersection of diagonals and M, N are midpoints of AB, CD respectively.

Problem 2. *In a competition, there are a contestants and b judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either pass or fail. Suppose k is a number such that, for any two judges, their ratings coincide for at most k contestants. Prove that*

$$\frac{k}{a} \geq \frac{b-1}{2b}.$$

This is a problem which requires enumerating skill. And counting is the only way for solution. So all of us got similar solutions.

The main idea is counting the total number of agreements in two different ways, first via judges and then via contestants. Denote the judges by J_i, $i = 1, 2, \ldots, b$ and the contestants by C_r, $r = 1, 2 \ldots, a$. Let k_{ij} be the number of agreements judges J_i and J_j have; there are $\binom{b}{2}$ such k_{ij}'s and since any two judges have atmost k agreements the average of the k_{ij}'s is not greater than k, i.e., $k \geq \frac{k_{ij}}{\binom{b}{2}}$. On the other hand, if p_r is the number of 'pass' got by C_r then the contribution to the total number of agreements by the p_r judges who 'pass' C_r is $\binom{p_r}{2}$; similarly for the number, $a - p_r$, of 'fails' got by C_r. Thus, the contribution to the total number of agreements by C_r is $\binom{p_r}{2}+\binom{a-p_r}{2}$. Therefore we get,

$$k \geq . \frac{k_{ij}}{\binom{b}{2}} = \frac{\sum_{r=1}^{a}\binom{p_r}{2}+\binom{a-p_r}{2}}{\binom{b}{2}}$$

The required inequality now follows from this by using the following result: let l be an integer and $m+n=2l+1$; then

$$\binom{m}{2}+\binom{n}{2} \geq \binom{l}{2}+\binom{l+1}{2}$$

Problem 3. *For any positive integer n, let $d(n)$ denote the number of positive divisors of n (including 1 and n itself).*

Determine all positive integers k such that

$$\frac{d(n^2)}{d(n)} = k$$

for some n.

This is an interesting problem. Firstly, I noticed that k should be odd, and after getting first few values, I conjectured that k can take any odd value and it was true.

The proof was by induction. It was enough to show that each odd number k is expressible as

$$\frac{(4a_1+1)(4a_2+1)\cdot\ \cdot(4a_l+1)}{(2a_1+1)(2a_2+1)\cdot\ \cdot(2a_l+1)},$$

where $a_i \geq 0$.

It is easy to see that this function is multiplicative, i.e, if k can take values a and b, then k can take the value ab. Hence it requires to prove only for primes. If the prime is of form $p=4m+1$, then we choose $a_1=m$ and we prove that k can take the value p. If p is of form $4m+3$, then we write $p=2^r s-1$ where s is odd and $r \geq 2$. Then we choose $a_1=\frac{(3p-1)}{4}$. Therefore, $(2a_1+1)=\frac{3p+1}{2}=p_1(\text{say})$. Then we take $a_2 = \frac{(3p_1-1)}{4}$ and hence $(2a_2+1) = \frac{(3p_1+1)}{2} = p_2$. Similarly we choose $a_3, a_4, \ldots, a_r$. Then we get

$$\frac{(4a_1+1)(4a_2+1)\cdot\ \cdot(4a_r+1)}{(2a_1+1)(2a_2+1)\cdots(2a_r+1)} = \frac{p}{2s+1}.$$

But k can take the value $2s+1$ (by induction) and hence k can take the value p.

Problem 4. *Determine all pairs (a,b) of positive integers such that ab^2+b+7 divides a^2b+a+b.*

This was an easy problem. We have

$$b(a^2b+a+b) = a(ab^2+b+7)+(b^2-7a).$$

Since ab^2+b+7 divides L.H.S, we get ab^2+b+7 divides b^2-7a. If we consider three cases, $(i) b^2-7a>0, (ii) b^2-7a=0, (iii) b^2-7a<0$, then we get the solutions as $(a,b) = (11,1), (49,1), (7k^2, 7k)$, where $k \geq 1$.

Problem 5. *Let I be the incentre of triangle ABC. Let the incircle of ABC touch the sides BC, CA and AB at K, L and M, respectively. The line through B parallel to MK meets the lines LM and LK at R and S, respectively. Prove that $\angle RIS$ is acute.*

Since this is a geometry problem, we found this easy. Rishi Raj gave a wonderful proof using harmonic pencils. He took N as the midpoint of KM and proved that the triangle RNS is a self polar triangle w.r.t the incircle (i.e, R is the polar of NS, S is the polar of RN and N is the polar of RS). This implies that I is the orthocentre of triangle RNS in which $\angle RNS$ is obtuse. Hence $\angle RIS$ is acute.

I used the fact that $SBML$ and $RBKL$ are cyclic. Hence I got $RB\cdot RS = RM\ \ RL, SB\ \ SR = SK\ \ SL$. Therefore

$$(RI^2-r^2)+(SI^2-r^2) = RM\ \ RL + SK\ \ SL =$$

$$RB\cdot RS + SR\cdot SB = RS^2.$$

Hence $cos\angle RIS = \frac{RI^2+SI^2-RS^2}{2(RI)(SI)} > 0$. Thus $\angle RIS$ is acute.

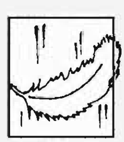
Problem 6. *Consider all functions f from the set* $\mathbf{N}$ *of all positive integers into itself satisfying*

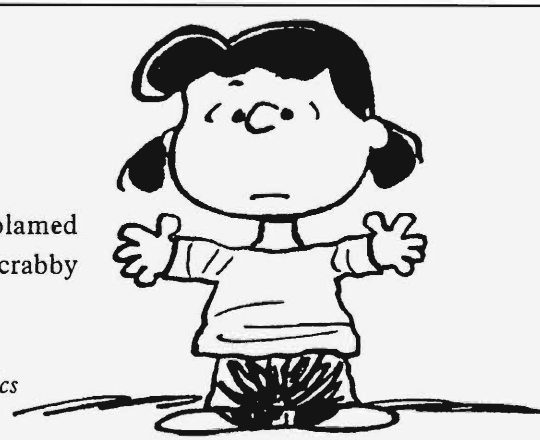
$$f\left(t^2 f(s)\right) = s\,(f(t))^2,$$

for all s and t in $\mathbf{N}$. *Determine the least possible value of* $f(1998)$.

This is a good problem and is treated as the toughest problem of this *IMO*. First, it is easy to see that $f(f(s)) = sf(1)^2$. Using this and manipulating the given equation we get that $f(1)f(t^2) = f(t)^2$. This implies that if prime p divides $f(1)$ then p divides $f(t)$ for all $t \geq 1$. It can be shown that $f(1)$ divides $f(t)$ for all $t \geq 1$. Then we define $g(t) = \frac{f(t)}{f(1)}$, which also satisfies the given conditions. We also have $gg(x) = x$ and g is multiplicative. Therefore g takes primes to primes. Since $1998 = 2\ \ 3^3\ \ 37$, we define $g(3) = 2, g(37) = 5$ which implies $g(2) = 3, g(5) = 37$ and $g(p) = p$ for all primes p other than $2, 3, 5, 37$. Hence we get that the least value of $g(1998) = 120$. And hence the least value of $f(1998) = 120$.

I think it was a wonderful performance by the Indians and we are happy that we were able to offer our humble Golden Tribute to India in the Golden Jubilee year of her Independence. I thank each and everyone who is directly or indirectly involved in this effort. I hope that we get six golds next year.





Can a person be blamed who is born with crabby genes.

From: *Gene Antics*