It was a great moment for Indians, when we got three gold and three silver medals in the 39th International Mathematical Olympiad, held in Taipei, Taiwan in July '98. It was well organised in almost all aspects. First of all, the problems in the final paper were of good standard. We had nice excursions, good cultural programmes, good guide,... etc., though the food situation was rather difficult for us, especially the vegetarians amongst us (Rishi Raj, Yogananda and I survived on fruits).

Our performance was really great. We once again proved our strength in Geometry. India was the only country to get all six full 7 points in three problems. After the first day’s examination was over in which all of us had got two problems right we were pleasantly surprised to hear our deputy leader, Yogananda, tell us that the second problem was a proposal from India! We, of course, did not know this. We are proud of R B Bapat, Indian Statistical Institute, Delhi, who was the author of this problem. More than 250 contestants got zero in that problem in which our team scored 42 out of 42! The one thing that is not in favour of us is that none of us could solve the 3rd and the 6th problem completely.

Here, I briefly discuss the problems which appeared in IMO:

Problem 1. In the convex quadrilateral $ABCD$, the diagonals $AC$ and $BD$ are perpendicular and the opposite sides $AB$ and $DC$ are not parallel. Suppose that the point $P$, where the perpendicular bisectors of $AB$ and $DC$ meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles $ABP$ and $CDP$ have equal areas.

Our team found this problem rather easy and altogether gave five different solutions, two using co-ordinate geometry, one using calculus and the other two being pure geom-
etry solutions. If the quadrilateral is cyclic, then it is easy to see that the triangles $APB$ and $CPD$ have same area. While proving the other way, I assumed that the result is not true and came to a contradiction. If the result is not true, then, assume $AP > CP$ (since they are not equal). We draw a circle with $P$ as centre and $PA$ as radius. Then extend the diagonals to intersect the circle again at $X$ and $Y$. Thus triangle $XPY = triangle APB = triangle CPD$ (this follows from the previous implication), which leads to a contradiction.

Chetan and Soham solved by proving the fact that $OMPN$ is a parallelogram, where $O$ is the point of intersection of diagonals and $M, N$ are midpoints of $AB, CD$ respectively.

**Problem 2.** In a competition, there are $a$ contestants and $b$ judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either pass or fail. Suppose $k$ is a number such that, for any two judges, their ratings coincide for at most $k$ contestants. Prove that

$$
\frac{k}{a} \geq \frac{b - 1}{2b}.
$$

This is a problem which requires enumerating skill. And counting is the only way for solution. So all of us got similar solutions.

The main idea is counting the total number of agreements in two different ways, first via judges and then via contestants. Denote the judges by $J_i$, $i = 1, 2, \ldots, b$ and the contestants by $C_r$, $r = 1, 2 \ldots, a$. Let $k_{ij}$ be the number of agreements judges $J_i$ and $J_j$ have; there are $\binom{b}{2}$ such $k_{ij}$’s and since any two judges have atmost $k$ agreements the average of the $k_{ij}$’s is not greater than $k$, i.e., $k \geq \frac{k_{ij}}{\binom{b}{2}}$. On the other hand, if $p_r$ is the number of ‘pass’ got by $C_r$ then the contribution to the total number of agreements by the $p_r$ judges who ‘pass’ $C_r$ is $\binom{p_r}{2}$; similarly for the number, $a - p_r$, of ‘fails’ got by $C_r$. Thus, the contribution to the total number of agreements
by $C_r$ is $\binom{p_r}{2} + \binom{a-r}{2}$. Therefore we get,

$$k \geq \frac{k_{ij}}{\binom{b}{2}} = \frac{\sum_{i=1}^{a} \binom{p_r}{2} + \binom{a-r}{2}}{\binom{b}{2}}$$

The required inequality now follows from this by using the following result: let $l$ be an integer and $m + n = 2l + 1$; then

$$\binom{m}{2} + \binom{n}{2} \geq \binom{l}{2} + \binom{l+1}{2}$$

**Problem 3.** For any positive integer $n$, let $d(n)$ denote the number of positive divisors of $n$ (including 1 and $n$ itself).

Determine all positive integers $k$ such that

$$\frac{d(n^2)}{d(n)} = k$$

for some $n$.

This is an interesting problem. Firstly, I noticed that $k$ should be odd, and after getting first few values, I conjectured that $k$ can take any odd value and it was true.

The proof was by induction. It was enough to show that each odd number $k$ is expressible as

$$(4a_1 + 1)(4a_2 + 1) \cdots (4a_l + 1)$$

$$(2a_1 + 1)(2a_2 + 1) \cdots (2a_l + 1),$$

where $a_i \geq 0$.

It is easy to see that this function is multiplicative, i.e, if $k$ can take values $a$ and $b$, then $k$ can take the value $ab$. Hence it requires to prove only for primes. If the prime is of form $p = 4m + 1$, then we choose $a_1 = m$ and we prove that $k$ can take the value $p$. If $p$ is of form $4m + 3$, then we write $p = 2^r s - 1$ where $s$ is odd and $r \geq 2$. Then we choose $a_1 = \frac{(3p-1)}{4}$. Therefore, $(2a_1 + 1) = \frac{3p+1}{2} = p_1$ (say). Then
we take \( a_2 = \frac{(3p_1 - 1)}{4} \) and hence \( 2a_2 + 1 = \frac{(3p_1 + 1)}{2} = p_2 \). Similarly we choose \( a_3, a_4, \ldots, a_r \). Then we get

\[
\frac{(4a_1 + 1)(4a_2 + 1) \cdots (4a_r + 1)}{(2a_1 + 1)(2a_2 + 1) \cdots (2a_r + 1)} = \frac{p}{2s + 1}.
\]

But \( k \) can take the value \( 2s + 1 \) (by induction) and hence \( k \) can take the value \( p \).

**Problem 4.** Determine all pairs \((a, b)\) of positive integers such that \( ab^2 + b + 7 \) divides \( a^2b + a + b \).

This was an easy problem. We have

\[
b(a^2b + a + b) = a(ab^2 + b + 7) + (b^2 - 7a).
\]

Since \( ab^2 + b + 7 \) divides L.H.S, we get \( ab^2 + b + 7 \) divides \( b^2 - 7a \). If we consider three cases, \((i)b^2 - 7a > 0, (ii)b^2 - 7a = 0, (iii)b^2 - 7a < 0,\) then we get the solutions as \((a, b) = (11, 1), (49, 1), (7k^2, 7k), \) where \( k \geq 1 \).

**Problem 5.** Let \( I \) be the incentre of triangle \( ABC \). Let the incircle of \( ABC \) touch the sides \( BC, CA \) and \( AB \) at \( K, L \) and \( M \), respectively. The line through \( B \) parallel to \( MK \) meets the lines \( LM \) and \( LK \) at \( R \) and \( S \), respectively. Prove that \( \angle RIS \) is acute.

Since this is a geometry problem, we found this easy. Rishi Raj gave a wonderful proof using harmonic pencils. He took \( N \) as the midpoint of \( KM \) and proved that the triangle \( RNS \) is a self polar triangle w.r.t the incircle (i.e, \( R \) is the polar of \( NS, S \) is the polar of \( RN \) and \( N \) is the polar of \( RS \)). This implies that \( I \) is the orthocentre of triangle \( RNS \) in which \( \angle RNS \) is obtuse. Hence \( \angle RIS \) is acute.

I used the fact that \( SBML \) and \( RBKL \) are cyclic. Hence I got \( RB \cdot RS = RM \cdot RL, SB \cdot SR = SK \cdot SL \). Therefore

\[
(RI^2 - r^2) + (SI^2 - r^2) = RM \cdot RL + SK \cdot SL = \n
RB \cdot RS + SR \cdot SB = RS^2.
\]

Hence \( \cos \angle RIS = \frac{RI^2 + SI^2 - RS^2}{2(RI)(SI)} > 0 \). Thus \( \angle RIS \) is acute.
Problem 6. Consider all functions $f$ from the set $\mathbb{N}$ of all positive integers into itself satisfying

$$f \left( t^2 f (s) \right) = s (f (t))^2,$$

for all $s$ and $t$ in $\mathbb{N}$. Determine the least possible value of $f(1998)$.

This is a good problem and is treated as the toughest problem of this IMO. First, it is easy to see that $f(f(s)) = sf(1)^2$. Using this and manipulating the given equation we get that $f(1)f(t^2) = f(t)^2$. This implies that if prime $p$ divides $f(1)$ then $p$ divides $f(t)$ for all $t \geq 1$. It can be shown that $f(1)$ divides $f(t)$ for all $t \geq 1$. Then we define $g(t) = \frac{f(t)}{f(1)}$, which also satisfies the given conditions. We also have $gg(x) = x$ and $g$ is multiplicative. Therefore $g$ takes primes to primes. Since $1998 = 2 \cdot 3^3 \cdot 37$, we define $g(3) = 2, g(37) = 5$ which implies $g(2) = 3, g(5) = 37$ and $g(p) = p$ for all primes $p$ other than 2, 3, 5, 37. Hence we get that the least value of $g(1998) = 120$. And hence the least value of $f(1998) = 120$.

I think it was a wonderful performance by the Indians and we are happy that we were able to offer our humble Golden Tribute to India in the Golden Jubilee year of her Independence. I thank each and everyone who is directly or indirectly involved in this effort. I hope that we get six golds next year.

Can a person be blamed who is born with crabby genes.

From: Gene Antics