Around Newton's Theorem

An Inverse Problem in Potential Theory

Cherian Varughese

A remarkable triumph of seventeenth century physics was the enunciation of the *law of gravitation* by Newton. In one bold step it unified phenomena as diverse as apples falling from trees and the motion of planets in their orbits.

Cherian Varughese is with the Indian Statistical Institute, Bangalore.

The profundity of the law is masked by the simplicity of its expression: Two point masses attract each other by a force whose magnitude is proportional to the product of their masses and varies inversely as the square of the distance between them. In symbols, if the location of point masses M and m are described by the vectors $\mathbf{x}$ and $\mathbf{y}$, respectively, in three dimensional space ($\mathbf{R}^3$), the mass m experiences a force given (in a suitable choice of units) by

$$\mathbf{F} = \frac{Mm}{\|\mathbf{x}-\mathbf{y}\|^2}\,\frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|}$$

where $\|\mathbf{x}-\mathbf{y}\|$ is the distance between the points $\mathbf{x}$ and $\mathbf{y}$ (see *Figure 1*).

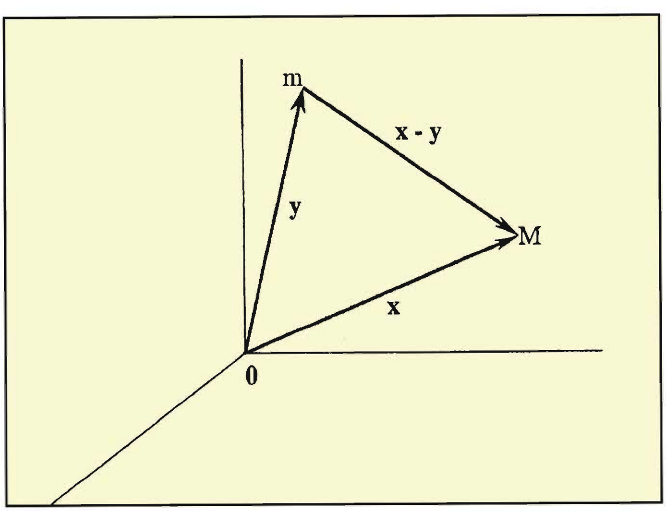

Figure 1.

We observe that the force is described by the expression

$$\mathbf{F} = m\left(-\nabla_{\mathbf{y}}\left(\frac{-M}{\|\mathbf{x}-\mathbf{y}\|}\right)\right)$$

where $\nabla_{\mathbf{y}}$ is the gradient with respect to the variable $\mathbf{y}$. Explicitly, if $\mathbf{y} = (y_1, y_2, y_3)$,

$$\nabla_{\mathbf{y}}\phi = \left(\frac{\partial\phi}{\partial y_1}, \frac{\partial\phi}{\partial y_2}, \frac{\partial\phi}{\partial y_3}\right) \in \mathbf{R}^3$$

We are thus led naturally to the notion of a potential function. Define the *potential* at the point $\mathbf{y}$ due to a mass M at the point $\mathbf{x}$ to be

$$\Phi_{M,\mathbf{x}}(\mathbf{y}) = \frac{-M}{\|\mathbf{x}-\mathbf{y}\|}$$

If we are dealing with an extended object and not a point mass, we can generalise the idea of a potential function to this case by considering infinitesimal elements of the object as point masses and summing the corresponding potentials. Thus, if the object occupies a region S in space with a density distribution given by the function ρ, the potential at a point $\mathbf{y}$ is given by

$$\Phi(\mathbf{y}) = \int_S \frac{-\rho(\mathbf{x})}{\|\mathbf{x}-\mathbf{y}\|}\, d^3\mathbf{x} \qquad (1)$$

In particular, if ρ is constantly 1 in S,

$$\Phi(\mathbf{y}) = \int_S \frac{-1}{\|\mathbf{x}-\mathbf{y}\|}\, d^3\mathbf{x}$$

The force exerted by the object on a point mass m located at $\mathbf{y}$ is given by

$$\mathbf{F} = -m\ \nabla\Phi(\mathbf{y})$$

An important observation is the fact that, outside S, Φ satisfies the Laplace equation

$$\frac{\partial^2\Phi}{\partial y_1^2} + \frac{\partial^2\Phi}{\partial y_2^2} + \frac{\partial^2\Phi}{\partial y_3^2} = 0$$

Newton's Theorem

This theorem is essentially a description of the potential due to a ball whose density is spherically symmetric, that is, the density function depends only on the radial distance from the centre of the ball.

If we are interested only in the potential outside the ball, we cannot distinguish it from a point mass located at the centre of the ball.If R is the radius of the ball centred at the origin

$$\Phi(\mathbf{y}) = \frac{-M}{\|\mathbf{y}\|} \quad \text{for every } \mathbf{y} \text{ with } \|\mathbf{y}\| \geq R$$

where M is the total mass of the ball.

That this is so can be seen by a simple computation in this case of the integral (1) that defines the potential.

In what follows, we will concentrate on what we will call *'homogeneous solids'*. Such a solid will be one which has the following properties:

(i) Compactness (i.e. the region that it occupies in space is closed and bounded).

(ii) Constant density equal to 1.

(iii) It is the closure of its interior. (This technical requirement essentially means that the object is the union of its interior and its boundary. For example, if $S = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1^2 + x_2^2 + x_3^2 \leq 1\}$, the interior is $\{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1^2 + x_2^2 + x_3^2 < 1\}$ and the boundary is $\{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$.) Further, we assume, for simplicity, that the volume of S equals the volume of its interior (i.e. we avoid the 'pathological' case where the boundary of S has positive volume.)

A Converse of Newton's Theorem

Having observed that the potential due to a homogeneous ball at points outside it is indistinguishable from that due to a point mass located at its centre, we can now ask the following question: If the potential outside a homogeneous solid is the same as that due to a point mass at some location, is the solid necessarily a ball?

The answer turns out to be that a homogeneous solid which is 'potentially a ball' is actually a ball. Although this fact is not hard to prove, it was discovered only as late as 1976 by Kondraskov. The proof described here is due to Zagier (see [1] for more details).

Theorem: Let S be a homogeneous solid such that

$$\Phi(\mathbf{y}) = \int_S \frac{-1}{\|\mathbf{x}-\mathbf{y}\|}\, d^3\mathbf{x} = \frac{-M}{\|\mathbf{y}-\mathbf{y}_0\|}$$

for every $\mathbf{y}$ in the complement of S where $\mathbf{y}_0$ is some fixed point and M is a constant. (It is not assumed that $\mathbf{y}_0$ lies in S.) Then S is a ball (and $\mathbf{y}_0$ is necessarily the centre of the ball).

Proof: By choosing the origin of our co-ordinates suitably we can, without loss of generality, assume that $\mathbf{y}_0 = \mathbf{0}$. Then our assumption becomes

$$\int_S \frac{1}{\|\mathbf{x}-\mathbf{y}\|}\, d^3\mathbf{x} = \frac{M}{\|\mathbf{y}\|} \quad \text{for every } \mathbf{y} \text{ that is not in } S. \qquad (2)$$

It can be shown that $\mathbf{0}$ is an interior point of S as follows. One can show that the left hand side of (2) is a continuous function of $\mathbf{y}$ on all of $\mathbf{R}^3$. This is easy but not trivial, and can be shown by using spherical polar co-ordinates on $\mathbf{R}^3$. If $\mathbf{0}$ is outside S or a boundary point of S, we can find a sequence of points outside S which converge to $\mathbf{0}$. The right hand side of (2) evaluated at these points would give a divergent sequence, which is a contradiction.

Taking gradients with respect to $\mathbf{y}$ we get

$$\int_S \frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^3}\, d^3\mathbf{x} = -\, M\, \frac{\mathbf{y}}{\|\mathbf{y}\|^3} \quad \text{for every } \mathbf{y} \text{ not in } S. \qquad (3)$$

Now take the inner product of both sides with $\mathbf{y}$ to get

$$\int_S \frac{\mathbf{x}\cdot\mathbf{y}-\|\mathbf{y}\|^2}{\|\mathbf{x}-\mathbf{y}\|^3}\, d^3\mathbf{x} = -\, \frac{M}{\|\mathbf{y}\|} \qquad (4)$$

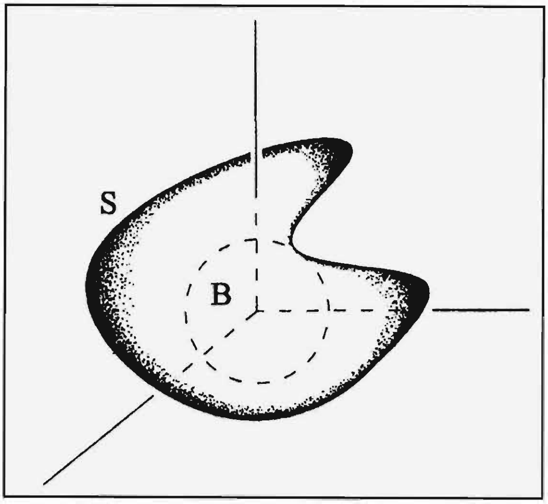

Figure 2.

Add (2) and (4) to get

$$\int_S \frac{\|\mathbf{x}\|^2 - \mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^3}\, d^3\mathbf{x} = 0 \quad \text{for every } \mathbf{y} \text{ not in } S. \qquad (5)$$

Let B be the largest closed ball centred at $\mathbf{0}$ and contained in S (see *Figure 2*). By repeating the above steps for B now, we get

$$\int_B \frac{\|\mathbf{x}\|^2 - \mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^3}\, d^3\mathbf{x} = 0 \qquad (6)$$

for every $\mathbf{y}$ not in B (hence also for every $\mathbf{y}$ not in S).

Subtracting (6) from (5) we get, for any $\mathbf{y}$ not in S,

$$\int_{S\backslash B} \frac{\|\mathbf{x}\|^2 - \mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^3}\, d^3\mathbf{x} = 0 \qquad (7)$$

(Here $S\backslash B$ is the collection of points that lie in S but not in B.)

From the continuity of the left hand side of (7) in $\mathbf{y}$, it follows that this equation is true even for any $\mathbf{y}$ that is a common point on the boundaries of S and B (such a point exists by the construction of B).

For such a choice of $\mathbf{y}$ and any $\mathbf{x} \in S\backslash B$,

$$\mathbf{x}\cdot\mathbf{y} \le \|\mathbf{x}\|\|\mathbf{y}\| < \|\mathbf{x}\|^2 \quad \text{(almost everywhere)}$$

Hence in (7) the integrand is positive (almost everywhere). From this and the assumptions on S, it is easy to see that the volume of $S \backslash B$ is zero and that $S \backslash B$ is actually empty. Therefore S is a ball.

A More General Question

At this point we make an extra assumption which, we must emphasize, is crucial to this section. We assume that the complement of S in three dimensional space is connected, i.e. any two points in the complement of S can be joined by a continuous curve lying entirely in the complement of S. It was emphasised earlier that the potential due to a solid S satisfies the Laplace equation in the region outside it. From this, it follows that the potential is *real analytic*[1] there. Since the complement of S is connected, the potential is completely determined by its values on any non-empty open set in the complement.

[1] A real analytic function f is one for which the Taylor series for it around any point p converges to f in a suitable neighbourhood of p. If f_1 and f_2 are two real analytic functions on a connected open set U agreeing on any non-empty open subset V, then $f_1 = f_2$ on U.

Hence, with the new assumption on S, we could have, in the previous section, made the weaker assumption that the potential due to S has the form $\frac{-M}{\|\mathbf{y}\|}$ for all $\mathbf{y}$ near infinity (i.e. outside some sufficiently large sphere). It would then follow that the potential everywhere outside S has this form and consequently S would have to be a ball.

This brings us to the following natural question:

If the potentials due to two solids S_1 and S_2 agree near infinity (i.e. they agree outside a sufficiently large ball), is $S_1 = S_2$?

The answer is yes if either of them is a ball (as we have seen already) and no in general. A counter example is the following:

We consider two spherical shells A_1 and A_2 and two balls B_1 (concentric with A_1) and B_2 (concentric with A_2) satisfying the following conditions:

(i) Volume of A_1 = Volume of B_1 (ii) Volume of A_2 = Volume of B_2 (iii) B_1 does not intersect A_2 and B_2 does not intersect A_1. (iv) A_1 and A_2 intersect in such a way that removing

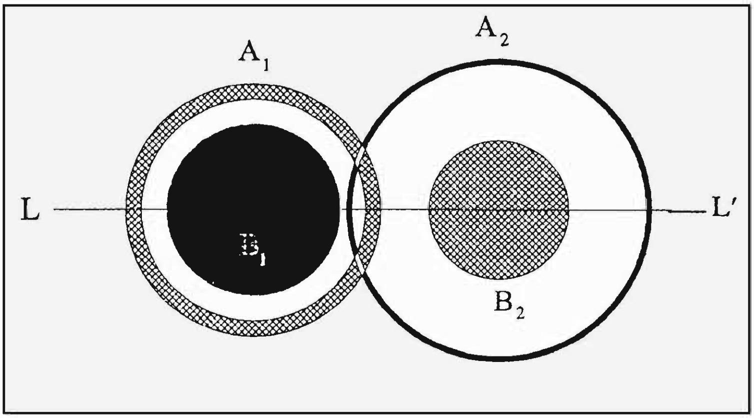

Figure 3.

$A_1 \cap A_2$ from A_1 (resp. A_2) leaves A_1 (resp. A_2) disconnected.

Figure 3 shows a plane section of a typical configuration.

We then choose

$$S_1 = (A_1 \cup B_2) - (A_1 \cap A_2)$$

$$S_2 = (A_2 \cup B_1) - (A_1 \cap A_2)$$

Thus S_1 and S_2 are solids obtained by revolving the hatched and darkened regions, respectively, about the axis LL'.

A simple application of Newton's theorem described earlier shows that although S_1 and S_2 are distinct, the potentials due to them agree near infinity.

In the above counter example, S_1 and S_2 each has three components. It can be modified to give a counter example in which S_1 and S_2 are connected and simply connected. (It is left as an interesting and challenging problem to the reader to make the necessary modifications!)

Some Related Ideas

The above considerations are representative of a more general class of questions which we may collectively call 'reconstruction problems'. This collection of questions can be crudely described as follows: Given some analytic data metric or topological information about S? An attempt is made below to illustrate these admittedly diffuse statements by means of examples. (In the situation discussed in the major part of this article, the information provided was some behaviour of the solution to the Laplace equation outside S. The attempt was then to recapture S.)

In what follows, we consider, for simplicity, *the plane* rather than three dimensional space.

In our first example, the set S is a region in the plane enclosed by a simple, closed curve γ parametrised as $(\gamma_1(t), \gamma_2(t))$. Here, γ_1 and γ_2 are 2π-periodic functions. By expressing the length, L, of the curve γ and the area, A, enclosed by it in terms of the fourier coefficients of γ_1 and γ_2, we get the *'isoperimetric inequality'*

$$L^2 - 4\pi A \geq 0$$

with equality in the above if and only if γ is a circle (i.e. S is a disc). For more details, see [2].

The second example is set in the context of what is traditionally called the Pompeiu problem. We say a bounded, measurable set S with positive measure (i.e. positive area) in the plane has the *Pompeiu property* with respect to a subgroup G of the group of rigid motions of the plane (combinations of rotations and translations) if the following holds: There is no non-trivial continuous function f on the plane such that $\int_{\sigma(S)} f = 0$ for every σ in G. Some of the non-trivial facts known in this regard are the following (see [3]):

(i) No bounded, measurable set S of positive area has the Pompieu property with respect to the *subgroup G of translations of the plane.*

(ii) Let S be a bounded, simply connected domain with piecewise smooth boundary. Assume S has the Pompeiu property with respect to the *full group of rigid motions of the plane.* Then the diffraction pattern produced on a screen *at infinity* by a parallel beam of monochromatic light, with S as the aperture, does not have a circle of zero illumination. (The beam is orthogonal to the plane of S and the screen is parallel to the plane of S.)

(iii) Assume that S is a bounded, simply connected region with piecewise smooth boundary. If S does *not* have the Pompeiu property with respect to the *full group of rigid motions*, then the boundary of S has no corners. *In fact, it is an open problem whether, in this situation, S is a disc.*

Finally, we turn to an example which was introduced in a paper by Mark Kac [4] amusingly titled *'Can one hear the shape of a drum?'* Mathematically, this question translates to the following: Let S be a region in the plane (not necessarily simply connected, i.e. S may have holes) with a suitably nice boundary. If we are given all the eigenvalues of the Laplacian on S that correspond to eigenfunctions which vanish on the boundary (Dirichlet boundary conditions), can we describe S completely?

From the eigenvalues we can gather at least some information about the region S. This includes the number of holes in S, the area of S and the perimeter of S. The details are presented in [5]. However, as was shown there, it turns out that there are non-isometric regions in the plane with the same set of eigenvalues. (We say two regions in the plane are non-isometric if there is no combination of rotations, translations and reflections that we can apply to one region to get the other.)

The ideas described above are not very far removed from those used in tomography. In the latter case, the goal is to reconstruct a function (the density of tissue, the density of some chemical element, etc.) that vanishes outside a bounded set. A CT-scan or a magnetic resonance image gives some (integrated) information about the function. The analogy with the cases described in this article becomes complete if we observe that recovering the set S corresponds to reconstructing a certain function, namely, the characteristic function of S.

Suggested Reading

[1] L Zalcman. Some inverse problems of potential theory, in Contemporary Mathematics. *American Mathematical Society*. Vol.63, 337-350, 1987.

[2] A Sitaram, The isoperimetric problem, *Resonance*. Vol. 2.No.9, 1997.

[3] S C Bagchi and A Sitaram. The Pompeiu problem revisited. *L'Enseignement Mathematique*. Vol. 36. 67-91,1990.

[4] M Kac, Can one hear the shape of a drum? *American Mathematical Monthly*. Vol. 73. 1-23, 1966.

[5] S Kesavan, Listening to the shape of a drum. *Resonance*. Vol. 3. Nos. 9 and 10, 1998.

Address for Correspondence
Cherian Varughese
Indian Statistical Institute
Bangalore , India