In this section of Resonance, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. “Classroom” is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

‘Puzzling Rectangles Revisited’

The problem, Puzzling Rectangles, proposed by B V Rajaram Bhat is more than a decade old and has several solutions. The solution given earlier in Resonance is perhaps the shortest, although it may be described as non-elementary, since it involves (double) integration. In this article, we recall a solution using graph theory. (In fact, Stan Wagon in his article [1] gives as many as fourteen proofs, including these two.) Graph-theoretic solutions of combinatorial problems (as also problems from other fields) are becoming more and more popular in recent times as they are more ‘transparent’ in nature.

A graph is a collection of points, called vertices, and lines joining some pairs of vertices, called edges such that between any two of the vertices there is at most one edge joining them. A bipartite graph is one in which the set of vertices can be partitioned into two subsets such that each of the edges has one end point in each of the subsets.

Coming to the problem at hand, choose the usual coordinate system and place the given rectangle so that its left-bottom corner coincides with the origin and the sides of the rectangle containing that corner are along the axes. By a lattice point we mean a point both of whose coordinates are integers. With any given tiling of the rectangle we shall associate a bipartite graph as follows. Let

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1 Question: Suppose that we have a rectangle which has been partitioned into sub-rectangles (see the figure) in such a way that every sub-rectangle in the partition has at least one side of integer length. Then show that the big rectangle also has at least one side of integer length. (See Mathematical Intelligencer. Vol.19, No.1, 1997 for related problems).

2 See page 87 in the December, 1997 issue of Resonance.
Suggested Reading


Let $A$ be the set of all (rectangular) tiles in the given tiling and $B$ be the set of all those lattice points which occur as corners of the tiles in the tiling. Notice that both $A$ and $B$ are non-empty — $A$ is clearly non-empty (unless the given rectangle has zero area!) and the origin belongs to $B$. We join a vertex $t$ in $A$ to a vertex $p$ in $B$ if (and only if) $p$ is a corner of $t$. For example, the tile which is at the left-bottom corner of the rectangle is joined to the origin. (Exercise: Tile a rectangle as per the conditions of the problem and draw the corresponding graph.) If we can prove that one of the three corners of the rectangle other than the origin is a lattice point we will be done. This is what we are going to prove.

Observe that a tile may not have lattice point corners even though its length or breadth is an integer (e.g. shift the unit square with origin as one of the corners slightly along the diagonal). The important thing to be noticed is that each tile has an even number (0, 2 or 4) of lattice points as corners (why?). Therefore, the total number of edges emanating from the set $A$ is even. On the other hand, any vertex in $B$ which is not a corner of the given rectangle is the corner of an even number of tiles (again, why?). Now, an even number of edges come out of $A$ and the origin is joined to only one vertex (=tile) of $A$. Hence we conclude that, to account for the even number of the edges, one more corner of the rectangle other than the origin has to be joined by a single edge to some tile, implying that it is a lattice point as desired.

Here are two more combinatorics problems for the readers to try and get graph-theoretic solutions. At first they may not look like having anything to do with graph theory but, in fact, do admit very nice graph-theoretic solutions.

1. From a committee of $n$ members $n$ distinct subcommittees are formed. Show that there is a member of the committee such that if she resigns from all the subcommittees on which she is serving, the resulting subcommittees (one of them may possibly be empty) are also distinct.

2. In an $n \times n$ array of real numbers ($n \geq 2$) the sum of the two largest numbers in any row is a constant, say, $\alpha$ and the sum of the two largest numbers in any column is a constant, say, $\beta$. Show that $\alpha = \beta$. 