Listening to the Shape of a Drum

2. You Cannot Hear the Shape of a Drum!

S Kesavan

In this concluding part, we will discuss the geometrical information which can be obtained from the eigenvalues of the Laplace operator and also describe a counter-example which led to the settling of the question raised by M Kac which was posed in the first part.[1]

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[1] Part 1 'The Mathematics of Vibrating Drums' appeared in *Resonance*, Vol.3, No.9, 1998.

In the first part of this article, we saw how the vibration of a body could be described via the solution of an initial-boundary value problem for the wave equation. The method of separation of variables for the wave equation led to the eigenvalue problem for the Laplace operator:

$$\left.\begin{array}{rcll} \Delta w + \lambda w & = & 0 & \text{in } \Omega \\ w & = & 0 & \text{on } \partial\Omega \end{array}\right\} \qquad (1)$$

where Ω is a bounded domain, *i.e.* a bounded and connected open subset of the Euclidean space $I\!R^N$ representing the region occupied by the body. (Thus, we will only consider bounded domains.) The Laplace operator has a sequence of positive eigenvalues $\{\lambda_n\}$ tending to infinity and these are the squares of the frequencies of vibration; the corresponding eigenfunctions $\{w_n\}$ give the normal modes of vibration. The question posed by Kac was whether the eigenvalues completely characterize the domain in question. We will now see to what extent this is true.

What Can you Hear?

In the one dimensional case, even the first eigenvalue (π^2/L^2) determines the length L of the vibrating string (assuming, of course, that all its other physical characteristics are known). In higher dimensions, it is indeed possible to get a lot of geometric information about the domain from the eigenvalues of the Laplacian.

We can, for instance, 'hear' the area (volume, in dimensions $N \geq 3$) of the domain. This is a consequence of a beautiful asymptotic formula for the eigenvalues, proved in 1911 by H Weyl. It states that

$$\lambda_n \sim 4\pi^2 \left(\frac{n}{B_N V} \right)^{\frac{2}{N}} \quad \text{as } n \to \infty. \tag{2}$$

(This means that, for large n, the ratio of the two terms on either side of the $\sim$ sign is very close to unity.) Here V is the volume of the domain and B_N is the volume of the unit ball in $\mathbb{R}^N$. For example, $B_2 = \pi$ and $B_3 = 4\pi/3$. (Do you know the formula for B_N? See*Box 1.*)

Thus, if two domains are isospectral, they have the same volume.

Box 1. The Volume of the Unit Ball

A simple application of the well known Gauss' divergence theorem to the vector $(x_1, \ldots, x_N) \in \mathbb{R}^N$, shows that if B_N is the volume of the unit ball, then its surface area is given by NB_N. Thus if we have a ball of radius r, its surface area will be $NB_N r^{N-1}$. Consider the N-fold integral

$$I_N = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-(x_1^2+\cdots+x_N^2)} \mathrm{d}x_1 \cdots \mathrm{d}x_N.$$

Clearly, $I_N = I_1^N$. Setting $r^2 = x_1^2 + \cdots + x_n^2$, we can evaluate I_N by polar coordinates.

$$I_N = \int_0^{\infty} e^{-r^2} NB_N r^{N-1} \mathrm{d}r.$$

If $N = 2$,

$$I_1^2 = I_2 = \pi \int_0^{\infty} e^{-r^2} 2r\mathrm{d}r = \pi.$$

Thus $I_1 = \sqrt{\pi}$ and $I_N = \pi^{N/2}$. Hence

$$\pi^{N/2} = NB_N \int_0^{\infty} e^{-r^2} r^{N-1} \mathrm{d}r = \frac{N}{2} B_N \int_0^{\infty} e^{-s} s^{(N/2)-1} \mathrm{d}s.$$

The integral on the extreme right is the Γ-function. Thus,

$$B_N = \frac{2\pi^{N/2}}{N\Gamma(N/2)}.$$

Note that $\Gamma(1/2) = \sqrt{\pi}$ by a simple change of variable and , by integration by parts, we have $\Gamma(p+1) = p\Gamma(p)$ and so $\Gamma(N/2)$ can be easily evaluated.

With the Weyl formula as the starting point, Pleijel obtained an improved asymptotic formula in 1954 for plane domains. If A is the area of a plane domain and L its perimeter, he showed that

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{A}{2\pi t} - \frac{L}{4}\frac{1}{\sqrt{2\pi t}} \text{ as } t \to 0.$$

Thus if two plane domains are isospectral, they have the same perimeter as well. By approximating a domain from inside by convex polygons, Kac proved that you can hear the area and the perimeter of convex domains by methods which were simpler than those of Weyl and Pleijel.

For simply connected domains, *i.e.* domains without holes, Pleijel showed that the asymptotic formula given above can be further refined by adding a term 1/6 on the right-hand side. Kac conjectured that if the domain had r holes, then the asymptotic formula should read

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{A}{2\pi t} - \frac{L}{4}\frac{1}{\sqrt{2\pi t}} + \frac{1}{6}(1-r) \text{ as } t \to 0.$$

This was proved by McKean and Singer in 1967. Thus, we can also 'hear' the number of holes a drum has! We can also obtain information on the curvature of the boundary.

However, all this is still insufficient to prove the isometry, *i.e.* congruence, of isospectral domains.

You Can Hear a Circular Drum!

The observations made hitherto lead to a proof that discs in the plane are uniquely determined by the eigenvalues of the Laplacian *i.e.* if Ω_1 is a disc and Ω_2 is isospectral to it, then Ω_2 will necessarily be a disc of the same radius.

The classical isoperimetric inequality states that for any plane domain of area A enclosed by a simple closed curve of length L, we have

$$L^2 \geq 4\pi A$$

with equality only for discs. (For a simple proof of this inequality, using Fourier series, see the article entitled *The*

Isoperimetric Problem by A Sitaram in the *Classroom* section of *Resonance*, Vol. 2, No. 9, September, 1997.) Hence, since the eigenvalues determine A and L, if $L^2 - 4\pi A = 0$, it will follow that the domain is a disc. So we can hear the shape of a circular drum! (Using other isoperimetric inequalities, one can prove the same result for spheres in higher dimensions. For more details, see *Box 2.*)

Answers to Kac's Question

Kac himself believed that the answer to his question would be negative but there was no way he could be certain. We will now briefly outline the developments in this direction leading to the final answers.

Even before Kac posed the question for plane domains, Milnor had shown, in 1964, the existence of two isospectral, but non-isometric, 16-dimensional tori. Thus, for compact manifolds (at least in 16 dimensions!), the question had been answered negatively. Milnor reduced his problem to one in number theory which had a well-known solution. After a fairly long lull, his method was taken up by Marie-France Vignéras and A Ikeda, who produced more examples in 1978. Vignéras gave examples of isospectral, non-isometric manifolds in all dimensions $N \geq 2$ and her 3-dimensional examples were quite remarkable in that even the topology of the manifolds (*i.e.* their fundamental groups) were different. She did show, however, that in 2-dimensions, isospectral tori were isometric (you can hear the shape of your medhu-vada or donut!).

The method of constructing counter-examples hitherto depended on finding some exotic coincidences with some other classical problems whose solutions were known. Thus, the research seemed to be leading mathematicians into esoteric subjects in search of such coincidences. The situation changed dramatically in 1985. The Japanese mathematician T Sunada proved a remarkable theorem wherein he reduced the problem of constructing isospectral manifolds to a straightforward problem in the theory of finite groups. This problem in group theory was well understood and led to a systematic way of constructing isospectral manifolds which were not

Box 2. Isoperimetric Inequalities

If a simple closed curve of length L encloses an area A, then the classical isoperimetric inequality

$$L^2 \geq 4\pi A$$

(with equality only for the disc) solves a problem known from the time of the Greeks as Dido's problem: Given a fixed perimeter L, what is the shape of the domain which has maximum area? The answer is the disc and only the disc. Equivalently, given an area A the disc minimizes the perimeter. This inequality can be extended to higher dimensions. If a bounded domain Ω has volume $|\Omega|$ and surface area $|\partial\Omega|$, then

$$|\partial\Omega| \geq N B_N^{1/N} |\Omega|^{1-(1/N)}$$

with equality only for the ball. Thus of all domains of fixed volume, the ball has the least surface area. (Does this explain why soap bubbles are round?) In $\mathbb{R}^3$, if V is the volume and S is the surface area, then

$$S^3 \geq 36\pi V^2.$$

By an isoperimetric problem, we now mean a problem of optimizing a domain dependent functional keeping some geometric parameter of the domain fixed. For example, we can ask: "Of all domains of fixed volume, which one has the least fundamental frequency?" This was posed by Lord Rayleigh in 1894 for plane domains (drums) and was solved independently by Faber and Krahn a quarter of a century later. Thus, if Ω^* is a ball whose volume is the same as that of Ω, the Rayleigh–Faber–Krahn isoperimetric inequality reads

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$$

with equality only for a ball, where λ_1 stands for the first eigenvalue of the Laplacian in the relevant domain. Hence, if a domain Ω is isospectral with a ball, it automatically has the same volume (by Weyl's formula) and so equality is attained in the above isoperimetric inequality. So Ω has to be a ball. You can hear the spherical shape in all dimensions!

the same. In the words of Robert Brooks, one of the important contributors to this subject, the search for isospectral manifolds at this stage became fairly simple, and actually, fun.

As far as domains in $\mathbb{R}^N$ were concerned, the French mathematician Bérard found, in 1980, some domains in $\mathbb{R}^N$ for which he could explicitly calculate all the eigenvalues of the Laplacian. Using his work, Urakawa, in 1982, constructed isospectral and non-isometric domains in $\mathbb{R}^N$ for all dimensions $N \geq 4$.

A final negative answer to Kac's question on plane domains came in 1992. Peter Buser had constructed, in 1987, some simple examples of isospectral surfaces which were distinct, using Sunada's theorem. Analysing his examples, and by 'mathematically flattening out' those surfaces, Carolyn Gordon, David Webb and Scott Wolpert found examples of isospectral domains in $\mathbb{R}^2$ which were not congruent. One such example is shown in *Figure 1.* (The results obtained by Gordon *et al* were announced in the *Bulletin (New Series) of the American Mathematical Society*, Vol. 27, No. 1, July 1992, pp. 134–38, and their detailed article appeared under the title "Isospectral plane domains and surfaces via Riemannian orbifolds" in *Inventiones Mathematicae*, Vol. 110, pp. 1–22, 1992.)

Some Mathematical Origami

Looking at *Figure 1*, we observe that the two domains have the same area and perimeter. In fact the two domains have some kind of resemblance; both are made up of 7 identical triangles.

While Gordon *et al* proved their isospectrality by sophisticated techniques based on Sunada's theorem, Pierre Bérard gave a more down-to-earth proof of the same by another method which consists of a 'cutting- and - pasting' approach towards the construction of an eigenfunction of one domain starting from an eigenfunction of the other, both of them having the same eigenvalue. This will prove that the domains are isospectral.

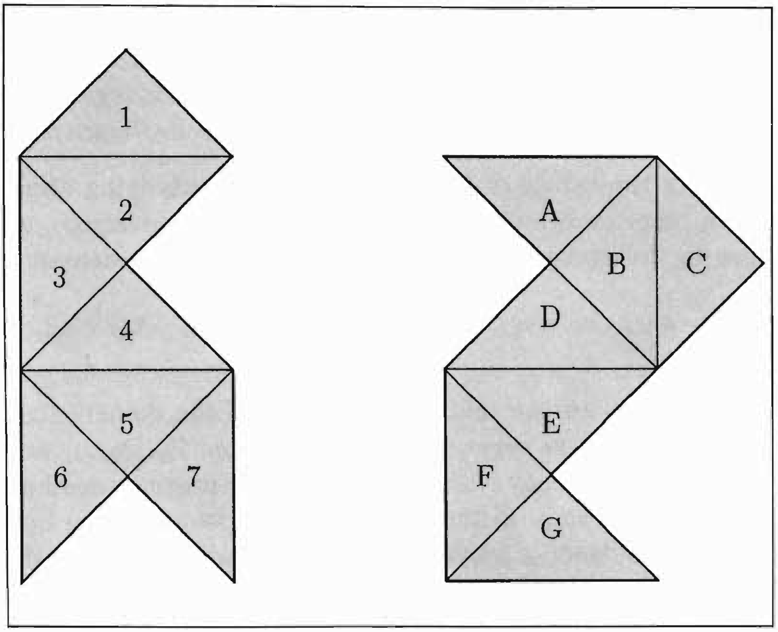

Figure 1. Two isospectral domains.

Let us label the 7 regions making up the domains as shown in *Figure 1*. Let w be an eigenfunction of the domain on the left, which we will call the first domain. Let the restriction of w to the region i be called w_i for $1 \leq i \leq 7$. Let $\overline{w_i}$ denote the reflection of w_i about the axis of symmetry of triangle i. We now define a function v on the second domain via its restrictions to the various regions as follows:

$$v_A = w_1 - \overline{w_2} - \overline{w_7} \quad v_B = -\overline{w_1} - \overline{w_3} - \overline{w_5}$$
$$v_C = -\overline{w_2} + w_3 - \overline{w_4}$$
$$v_D = w_1 - \overline{w_4} - \overline{w_6} \quad v_E = w_2 - \overline{w_5} + w_6$$
$$v_F = w_3 - \overline{w_6} - \overline{w_7}$$
$$v_G = w_4 - \overline{w_5} + w_7$$

Then v will be an eigenfunction of the second domain corresponding to the same eigenvalue.(The above formulae mean that, in order to get the value of v in a given triangle, we add the values of the functions at the corresponding points of the relevant triangles of the first domain as indicated.)

We can describe this procedure via paper folding. Assume that we have a cut-out of the first domain. We label the reverse sides of the triangles by $\overline{1}$ through $\overline{7}$. If we fold

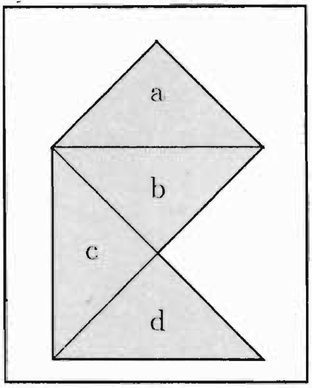

Figure 2. Folded domain.

along, say, the line between triangles 4 and 5 so that the region 3 is covered by $\overline{6}$ and 4 is covered by $\overline{5}$, and then along the line between triangles 5 and 7 so that region $\overline{5}$ is now covered by 7, we get the domain shown in *Figure 2.*

Adding the values of the function w at points lying above each other (subtracting, if the paper has been reversed), we get the following transposed function on this new domain:

$$u_a = w_1 : u_b = w_2 : u_c = w_3 - \overline{w_6} : u_d = w_4 - \overline{w_5} + w_7.$$

In this way we can fold several copies of the domain (call it Ω) in different ways to get new domains $\Omega_1, \Omega_2, ..$ and transposed versions of the function w. By pasting together, in a suitable way, all these new domains, we can get a new domain Ω^* and a transposed function, which will be the sum of the individual transpositions, and which will be an eigenfunction for Ω^*, with the same eigenvalue.

Evidently we cannot paste as we like. For instance, to start with, the new domain must also be made up of the same number of triangles (7, in our example). In order that the new transposed function is an eigenfunction, we need to ensure two things: that the new function satisfies both parts of the equation (1). Indeed, the function will certainly satisfy the differential equation in the interior of each triangle. But there could be trouble across the seams. From the regularity theory of elliptic partial differential equations, of which (1) is an example, we can avoid the above mentioned trouble if we ensure that the transposed function is continuous and has continuous first order derivatives across the boundaries of the folded and pasted triangles.

The example of *Figure 1* is one in which three such folded domains are glued together, the first being the folding described via *Figure 2.* Several possibilities exist. A systematic way of doing this has been neatly described in an article by S J Chapman (see Suggested Reading). So we can actually go ahead and construct isospectral domains with some paper and scotch tape.

Winding Up

We have seen that, as expected by Kac himself, his question has been answered negatively. However, this is not the end of the story and a lot of questions remain as to the connection between the spectrum of the Laplacian and the geometry of the domain. We saw that we can determine, from the spectrum, the area, the perimeter and also some information on the curvature of the boundary of the domain. Some topological features, such as the number of holes, can be recovered. In a fractal sense, we can also determine how 'wiggly' the boundary is, *i.e.* we can compute what is known as the Minkowski dimension of the boundary. We cannot hear the 'general topology' nor some more esoteric items such as the patterns of criss-crossing closed geodesics. Just how much geometrical information the spectrum holds, still remains to be seen.

Note: For more information on the physics of a vibrating drum, see box item on page 58.

Going back to the basic question, all the examples of isospectral domains constructed hitherto are polygonal, or more generally, have non-smooth boundaries. So one can still ask, "Are smooth isospectral domains isometric?" The answer is not yet known.

Suggested Reading

[1] M Kac. Can one hear the shape of a drum?. *American Mathematical Monthly*. Vol. 73. April, Part II. 1–23.1966.

[2] M H Protter, Can one hear the shape of a drum? Revisited. *SIAM Review*. Vol. 29. No. 2. 185–197, 1987.

[3] S J Chapman. Drums that sound the same. *American Mathematical Monthly*.Vol. 102. February, 1995. pp. 124–138.

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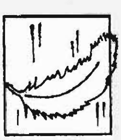


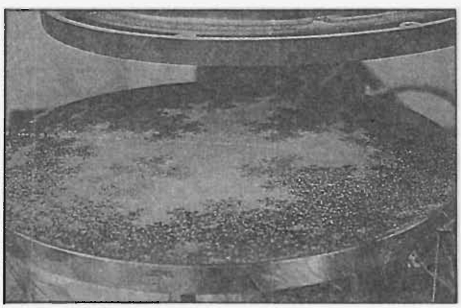
A mathematician confided
That a Mobius strip is one-sided,
And you'll get quite a laugh
If you cut one in half,
For it stays in one piece when divided.

-ANONYMOUS

Vibrations of a Fractal Membrane

We learn quite early in our education that the vibrational modes of a circular or a rectangular membrane are very different from the modes of vibration of a string. In particular the vibrations of a uniform membrane result in anharmonic overtones in contrast to the vibrations of a uniform string which always gives out harmonic overtones. In its fundamental mode of vibration it has maximum displacement at the centre. The displacement gradually goes to zero as the boundary is approached. Further, we also learn that in every mode of vibration of the membrane, nodal lines appear along which the membrane is actually at rest. If a fine powder is initially spread on the membrane then its particles will collect along these nodal lines.

What if the boundary of the vibrating membrane is not a circle or a rectangle but has jagged edges ? This question was answered in 1991 by B Sapoval and his co-workers. They worked with a plastic membrane stretched across a metal frame with a fractal boundary. The membrane was set in vibration with a loud speaker placed above the membrane. When a fine powder was sprinkled on the membrane the particles got agitated by the vertical motions of the membrane and settled in nodal regions. They found that the lowest frequency mode was much like that of a circular or a rectangular membrane. It had a vibration maximum at the centre and decayed rapidly as one moved towards the boundary. At higher frequencies the membrane behaved very differently. Instead of modes with nodal lines more complex resonances were observed. The peculiar thing noticed by them was the localization of vibration in a certain region of the membrane while elsewhere the membrane was practically at rest. In this region the particles of the powder will be jumping up and down and thus can scatter light . Therefore, the localized mode can be easily visualised by shining a laser beam on the membrane. The figure shows one such localized vibration. Another interesting observation pertains to the migration of this localized vibration. Generally after a finite time the localized mode gradually disappeared but only to appear in the diametrically opposite equivalent site on the membrane!



*Still photograph of a localized vibration. Exposure time was 1sec.**

This is also the right place to mention another important outcome of the analysis of the vibrations of membranes with fractal boundaries. Detailed calculations have established a conjecture of M V Berry that we can make out from the vibration spectrum whether the membrane has a fractal boundary or not.

Reference: Physical Review Letters **Vol.67,Page 2974 ,Year 1991,** Authors: B Sapoval, Th Gobron and A Margolina, Title: Vibrations of Fractal Drums.

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