Chaos

3. Studying Chaos in the Laboratory

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The non-linear differential equations used to study chaotic systems can often be simulated via electronic circuits. These circuits can be used effectively to demonstrate most features exhibited by chaotic systems. In this part we undertake an experimental study of chaos using electronic circuits.

Introduction

As pointed out in the first part[1] chaos can develop in systems obeying non-linear equations. But simple non-linear equations can have very complicated solutions. We consider an example here. Among the most famous examples of equations with chaotic attractors are the Lorenz and the Rössler systems. The equations that describe the Rössler attractor are 'nearly' linear –

$$\begin{aligned} \dot{x} &= -(y+z) \\ \dot{y} &= x + y/5 \\ \dot{z} &= 1/5 + z(x-c) \end{aligned} \qquad (1)$$

The solution of these differential equations for given initial conditions is a function of the parameter c. For $c \leq 2.5$, the solution is a limit cycle (*Figure 1*); between this value and 3.5, there is a period-doubling bifurcation: the limit cycle orbit becomes almost twice as long and the trajectory now goes through both these loops to complete one orbit (*Figure 2*). This *period doubling* continues as the parameter value is changed (see *Figure 3* for period 4), each doubling bifurcation occurring at a critical value of the parameter. The interval between successive bifurcations keeps decreasing and tends to a geometric progression as the number of bifurcations increases. This geometric progression is characterised by the Feigenbaum

[1] Previous articles of this series were:
1. Introduction to Chaos, *Resonance*, Vol.3, No.4, 1998.
2. Routes to Chaos, *Resonance*, Vol.3, No.6, 1998.

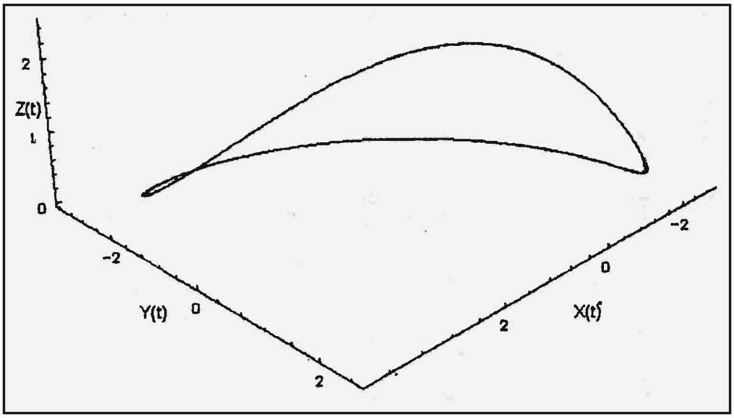

Figure 1. Period 1 orbit of the Rössler system.

constant, namely $\delta = 4.6692016...$, as in the case of the logistic map.

Solving Equations Using Op-Amps

Since equations such as the above can be simulated through electronic circuits this offers one method by which the phenomenology of chaos can be studied 'experimentally'.

The design of such an electronic circuit is motivated by the fact that op-amps (741, for example) (*Box 1*) can be used to integrate signals. The circuit shown in [2] *Figure c* is a summing amplifier that adds currents and voltages. The circuit shown in *Figure d* integrates the potential V at the input point 1. *Figure 4* shows

[2] See Box 1 for Figures *a–d*.

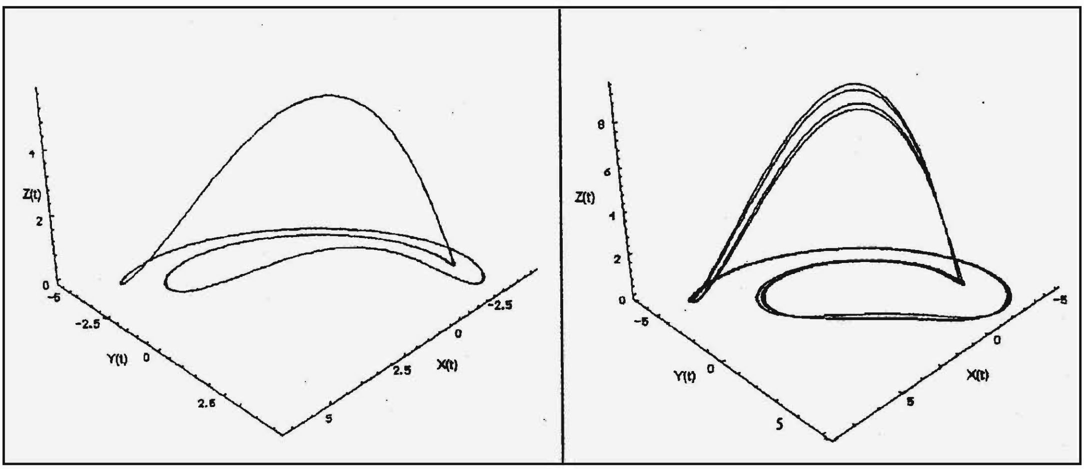

Figure 2 (left). Period 2 orbit of the Rössler system.

Figure 3 (right). Period 4 orbit of the Rössler system.

Box 1.

The *Operational Amplifier* (op-amp) is a commonly used high-gain amplifier. It usually has differential input *i.e.* its output is proportional to the difference of its two inputs (see *Figure a*). Terminal 1 is called the non-inverting terminal and terminal 2 the inverting terminal. Terminal 3 is for output. The other terminals are for the DC power supply $+V_{cc}$ and $-V_{ee.}$ The proportionality factor, denoted by A, is called the *open loop gain* and is typically very high (10^5 for the 741). It also has other characteristics desirable of a voltage amplifier – high input impedance, low output impedance, large band-width and stability with respect to temperature.

The op-amp is made up of several *direct-coupled* transistor amplifiers and is fabricated on an IC. For example the 741 is available as on a 8-pin plastic package. As the op-amp is made of transistors, the open-loop gain of the op-map depends on the values of the transistor parameters. These are temperatures sensitive, making A vary with temperature as well. Also, as the output is limited by $+V_{cc}$ and $-V_{ee}$, and the gain is very high, only very small signals can be amplified. These considerations make straightforward use of the op-amp impractical.

There is a very general technique which is used to overcome the aforementioned difficulties. It is called *negative feedback*. This involves feeding a signal proportional to the output (but of opposite sign to the input) back into the input. It is called negative because it tends to reduce the input. *Figure b* shows an op-amp with negative feedback. Under negative feedback the op-amp's gain reduces (allowing larger signals to be amplified) and the effective gain A_f becomes independent of the open-loop gain A to a very good approximation. There are a few beneficial side effects – the input impedance increases, output impedance decreases and bandwidth also increases.

There is also one more remarkable effect: the inverting and the non-inverting terminals have the *same* potential. Thus if the non-inverting terminal is grounded, the inverting terminal also has potential zero, even though it is not connected to the ground. Hence it is called the *virtual ground.*

Now consider the circuit of *Figure c*. Terminal 1 is virtual ground and so the current in the inverting terminal is $i = v_i/R_i$. As the input impedance is very large, most of this goes into the feedback impedance R_f. Again as Terminal 1 is virtual ground, the potential at the output is $-iR_f$. Combining we have,

$$v_o = -iR_f = -v_i(R_f/R_i) \Rightarrow A_f = (v_o/v_i) = -R_f/R_i$$

Thus the circuit works as an *inverting amplifier* of gain $-R_f/R_i$. If $R_f = R_i$, then it is just an inverter. If there are several inputs as in *Figure e*, the current in the feedback resistor is just the sum of the currents from all the inputs and thus the output is the inverted sum of the inputs. This circuit thus adds signals and is called a *summing amplifier*.

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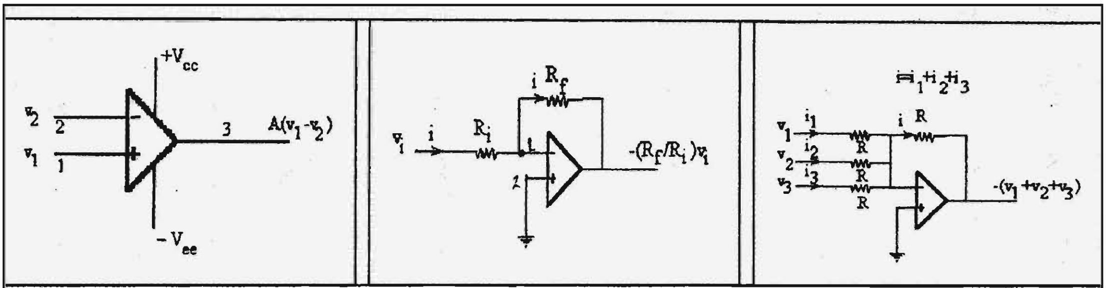


Figure a(left). The circuit representation of an op-amp.
Figure b(centre). The functioning of an op-amp.
Figure c(right). A summing amplifier.

An interesting circuit results if one replaces R_f in *Figure b* with a capacitor C, as shown in *Figure d*. Now the current in inverting terminal is v_i/R as before, and it still goes into the feedback impedance. But the potential at the output is decided by the charge on the capacitor ($v_o=Q/C$) which is the integral with respect to time of the current, thus the output is effectively the integral of the input.

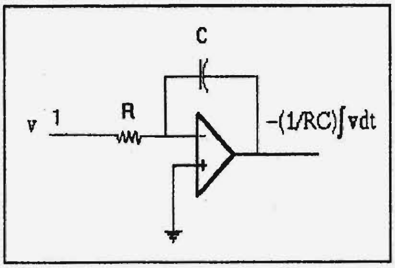


Figure d. Integrating a single signal using an op-amp.

$$i = dQ/dt$$
$$v_i/R = -C\,(dv_o/dt)$$
$$v_o = -1/RC \int v_i\,dt$$

This circuit is known as an *integrator* and is the bread and butter of differential equation solving.

Other circuits (differentiator, multiplier etc.) can similarly be constructed and combined to perform any computation. Such a circuit is known as an *analog computer*

the circuit that would solve the $\dot{x}$ part of the Rössler equations if y and z were to be given at points 1 and 2 in the circuit. One could build a circuit (see *Figure 5*) for the 2nd and the 3rd Rössler equations, the circuit for $\dot{z}$ utilizing a multiplier (AD632AD)

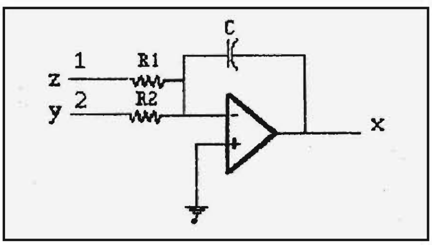
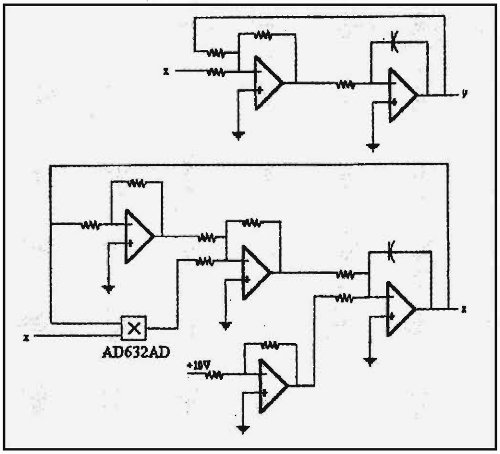


Figure 4. Solution of the $\dot{x}$ equation of the Rössler system.



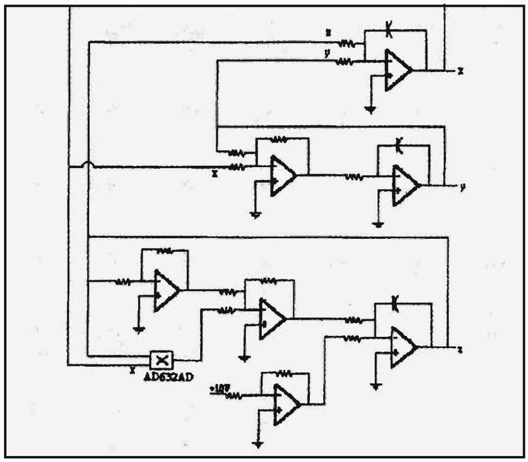


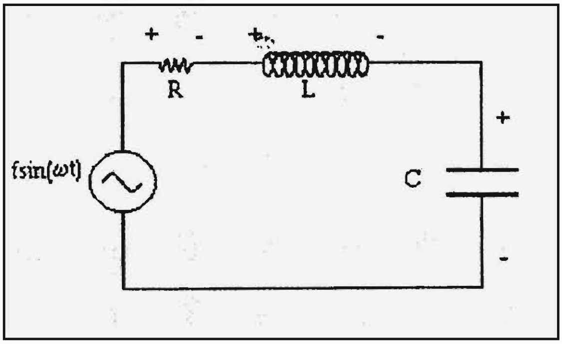
Figure 5(left). Circuit for solving the y and $\dot{z}$ equations of the Rössler system. Figure 6 (right). The complete circuit for solving the Rössler equations.

to produce the xz term in the equation. Now if one couples these circuits, as shown in *Figure 6* one obtains the circuit which solves the coupled equations in (1). The values of the resistors and the capacitors to be used would be decided by the values of the constants in the equation.

Analog simulations provide an inexpensive method for the qualitative study of differential equations. With the judicious choice of 'off-the-shelf' components, it is in principle possible to solve any set of differential equations by the analog procedure of constructing the equivalent circuit. In practice, though, it can be difficult to obtain the multiplier. For purposes of studying chaotic behaviour, several systems are known by now, and indeed there are circuits that have been designed specifically for the purpose, as for the example of Chua's circuit [1]. In the following section we describe one such system that is very similar to the Rössler set of equations, for which the equivalent circuit is easy to construct and study.

Analog Simulation of Chaos

For analog simulations, one can use either LCR (for a few select systems) circuits, or the more general method of op-amps. It is easy to see that a simple LCR circuit shown in *Figure 7* has the same form for the equation of state as the damped and driven pendulum (with sin x replaced by x).



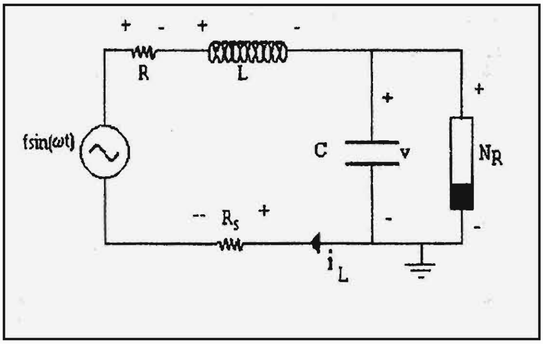


Figure 7(left). The LCR circuit.

Figure 8(right). The LCR circuit with a non-linear resistor, N_R.

$$\frac{d^2q}{dt^2} = \frac{1}{LC}q - \frac{R}{L}\dot{q} + \frac{f}{L}\sin\omega t. \qquad (2)$$

The LCR circuit of (2) could be made to simulate a chaotic system by introducing a non-linear element N_R as indicated on the circuit in *Figure 8* where R, C and L are normal linear components, but N_R is a non-linear resistor with the current-voltage characteristics shown in *Figure 9*.

$$\begin{aligned} I_R = g(v) &= m_0v - (m_1 - m_0)B_p, && v < -B_p \\ &= m_1 v, && -B_p \le v \le B_p \\ &= m_0 v + (m_1 - m_0)B_p, && v > B_p \end{aligned}$$

Here, m_1 and m_0 are the slopes of the two linear segments of the curve, $\pm B_p$ are the points at which the slope changes. This resistor is called Chua's diode and can be realised in the laboratory using op-amps. One can see that Kirchhoff's equations for the

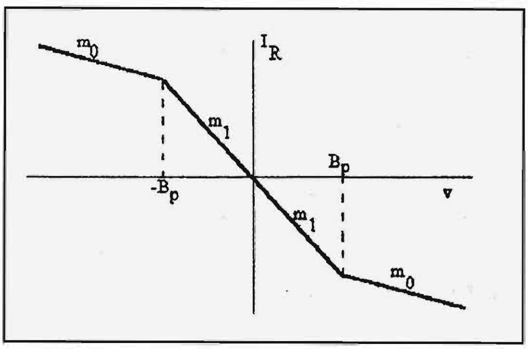


Figure 9. The current-voltage characteristics of the Chua diode.

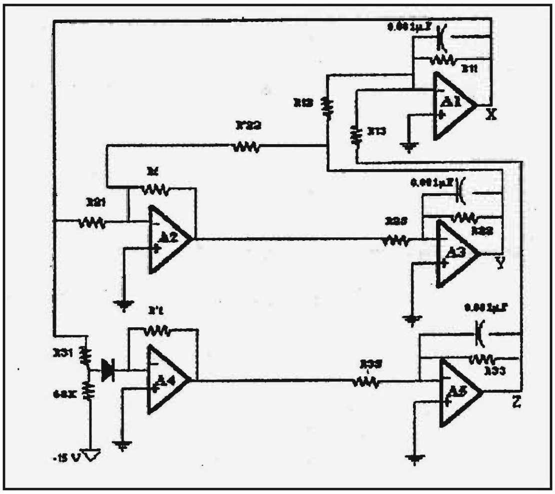


Figure 10. A circuit similar to the Rössler system.

circuit are,

$$C(\mathrm{d}v/\mathrm{d}t) = i_L - g(v)$$
$$L(\mathrm{d}i_L/\mathrm{d}t) = -Ri_L - R_s i_L - v + f\sin(wt)$$

The circuit exhibits chaotic dynamics for $m_1 = -0.76$ms, $m_0 = -0.41$ms, $B_p = 1.0$V, $C = 10$nF, L = 18 mH, $R = 1340\ \Omega$, $R_s = 20\Omega$ and $\nu = \omega/2\pi = 8890$ Hz. One can observe the attractor by plotting the potential difference across R_s (which is a measure of the current i_L) and the potential (v) across the capacitor, in the X-Y mode of the CRO. One can observe period doubling by varying the amplitude of the driving force.

The other method involves the use of operational amplifiers. We have already discussed the approach to building such a circuit for (1). Using the same line of argument one can see that the circuit shown in *Figure 10* integrates the equations given below, which mimic the Rössler equations.

$$\dot{x} = (\tau x + \beta y + \lambda z)\ ;\ \dot{y} = (-x - \gamma y + 0.02y)$$
$$\dot{z} = (-g(x) + z)$$

where $g(x) = 0$ if $x \le 3$ and $\mu\,(x-3)$ if $x \ge 3$.

Here, the values of the parameters are decided by the values of the resistances[2].

[2] Values of resistances, $\tau = r_{2S}/r_{11}$, $\beta = r_{2S}/r_{12}$, $\lambda = r_{2S}/r_{13}$, $r_f = r_{21}$, $\gamma = r_f/r'_{22}$, $r_{2S}/r_{22} = 0.02$, $r_{2S} = r_{33}$, $\mu = r'_f/r_{31}$ and $r_{3S} = r_{2S}$.

Typical values of these components are:

$r_{11} = 2\mathrm{M}\Omega$, $r_{12} = 200\mathrm{K}\Omega$, $r_{13} = 100\ \mathrm{K}\Omega$, $r_{33} = 100\mathrm{K}\Omega$, $r_{22} = 5\mathrm{M}\Omega$, $r_{3S} = 100\mathrm{K}\Omega$, $r'_{22} = 75\mathrm{K}\Omega$, $r_f = r_{31} = 10\ \mathrm{K}\Omega$, $r_{21} = 10\mathrm{K}$, $r'_f = 150\mathrm{K}$.

Any op-amp (for example 741) and diode can be used. The circuit shown is relatively simple to construct, and the different dynamical behaviours like periodicity, period doubling and chaos can be observe.

The use of electronic circuits as discussed above proves to be extremely useful while studying purely qualitative features of chaotic dynamics though, for a quantitative analysis, numerical simulations using computers are more convenient. One drawback of using electronic circuits is the lack of control and versatility. Fixing initial conditions and making measurements can prove difficult. Also, if one wishes to modify the equations being simulated, its back to the soldering iron! On the other hand, though, the study of dynamics via the procedure of constructing equivalent circuits provides different physical insight: laboratory experimentation gives a direct feeling for what chaos theory is all about.

Suggested Reading

[1] M P Kennedy. *Phil. Trans. Roy. Soc.* Series A, 353, 13–32, 1995.

[2] T L Carroll. *Am. J. Phys.* 63. 377, 1995.

Addendum

A few readers of Part 1 of this series have pointed out that in some places the text might be misunderstood. Hence, this addendum.

The introductory part may create an impression that approximate solutions are chaotic. We would like to stress that chaos is a feature of exact solutions obtained either numerically or analytically.

In the discussion on trajectories and orbits again the text might lead to some confusion. In this context we would like to emphasise that on a strange chaotic attractor, two trajectories will tend to diverge from each other exponentially if they come sufficiently close. As stated in *Box 2* the combined effects of dissipation and driving in the non-linear system gives the attractor a fractal geometry with a trajectory that does not intersect itself.

Incidentally, there is an inadvertent error in the legend to *Figures 2 and 3 of Part 1*. They are not for chaotic pendulums but are the outputs of the time series of an electronic circuit. In this sense, they rightly belong to this part of the series.

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