From Fourier Series to Fourier Transforms

Thanks to the pioneering work of Bernoulli, Euler, Lagrange and **Fourier**, today every student of science and engineering knows that a periodic function $f(x)$ of period 2π can be expanded (possibly in an infinite series) in terms of $\sin kx$ and $\cos kx, k = 0, 1, 2, \cdots$ (see [1].) For more on the historical developments that led to this idea, the reader is encouraged to consult [2]. Thus, using complex notation, a 2π-periodic function $f(x)$ on the real line $\mathbb{R}$ can be 'written' in the form

$$f(x) \approx \sum_{n=-\infty}^{\infty} a_n e^{inx} \qquad (*)$$

where, as Fourier observed, the a_n's are given by the formula $a_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\mathrm{d}x$. In what sense does the series on the right converge, and if it does converge, in what sense is it equal to $f(x)$? These are surprisingly subtle matters and the answers depend on the nature of the function $f(x)$. For instance, if, as is the case in most applications, f is square-integrable on $[0, 2\pi]$, i.e. $\int_0^{2\pi} |f(x)|^2\mathrm{d}x < \infty$, then the series on the right hand side of $(*)$ is absolutely square summable on $[0, 2\pi]$, i.e. $\sum_{n=-\infty}^{\infty} |a_n|^2 < \infty$, and converges in the mean square sense to f i.e. $\int_0^{2\pi} |f(x) - \sum_{k=-n}^{n} a_k e^{ikx}|^2\mathrm{d}x \to 0$ as $n \to \infty$. If f is reasonably smooth, then the series converges uniformly to f. For more on this fascinating story the reader is referred to [2].

The appearance of the function e^{inx} is not entirely accidental! Note that the interval $[0, 2\pi]$, with the end points identified, can be thought of as the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, via the identification $\theta \longleftrightarrow e^{i\theta}$. Under this identification, the function e^{inx} is really the function χ_n on S^1 defined by $\chi_n(z) = z^n$. S^1 is a group under multiplication and the χ_n' $s, n = 0, \pm 1, \pm 2, \cdots$ account for *all* the continuous homomorphisms of S^1 into S^1. We now pass on to the case of functions $f(x)$ defined on $\mathbb{R}$ which satisfy a reasonable condition like $\int_{-\infty}^{\infty} |f(x)|\mathrm{d}x < \infty$ (integrability) *or* $\int_{-\infty}^{\infty} |f(x)|^2\mathrm{d}x < \infty$ (square integrability). Notice that such functions (which are not identically zero) cannot be periodic.

(Why not?) So, no matter how we re-scale the real line, we cannot really hope to 'expand' such functions in terms of $\sin kx$ and $\cos kx, k = 0, 1, 2, \cdots$, as in the periodic case. Taking the cue from the fact that the functions e^{inx} correspond to the homomorphisms χ_n of S^1 into S^1, we look for continuous homomorphisms of $\mathbb{R}$ into S^1. It turns out that *all* such continuous homomorphisms of $\mathbb{R}$ into S^1 are given by $\varphi_\lambda(x) = e^{i\lambda x}, \lambda \in \mathbb{R}$. ($\mathbb{R}$ is a group with $+$ as the group law.) So we are tempted to conjecture that $f(x)$ can be 'expanded' in terms of $e^{i\lambda x}$. Clearly since we have a continuum of such functions, we may have to replace the infinite series in the periodic case by an integral in the non-periodic case. That this can be done is one of the great triumphs of Fourier transform theory. To cut a long story short, for an *integrable* function $f(x)$, define $\hat{f}(\lambda)$, the *Fourier transform* of $f(x)$, by $\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}\mathrm{d}x$. Then it turns out that $\hat{f}(\lambda)$ is a continuous function on $\mathbb{R}$, and $|\hat{f}(\lambda)| \to 0$ as $|\lambda| \to \infty$. If the function f is *sufficiently well behaved*, then we can recover $f(x)$ from $\hat{f}(\lambda)$ by the Fourier inversion formula $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\lambda)e^{i\lambda x}\mathrm{d}\lambda$. Moreover if f is *also square-integrable*, then we do have mean square convergence as in the Fourier series case, i.e. $\int_{-\infty}^{\infty} |f(x) - \frac{1}{2\pi}\int_{-R}^{R} \hat{f}(\lambda)e^{i\lambda x}\mathrm{d}\lambda|^2\mathrm{d}x \to 0$ as $R \to \infty$, and $\int_{-\infty}^{\infty} |f(x)|^2\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2\mathrm{d}\lambda$. Actually there is a purely *heuristic* way of justifying the Fourier inversion formula starting from the Fourier expansion of a periodic function. (See pages 204-205 of [3]).

There is another, perhaps intuitively more appealing way of motivating the Fourier transform. Notice that the Fourier inversion formula is an expansion of f in terms of the functions $\varphi_\lambda(x) = e^{i\lambda x}$. Now each φ_λ is an eigenfunction for the one-dimensional Laplace operator $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with eigenvalue $-\lambda^2$. Among these the 2π-periodic ones are precisely $e^{ikx}, k = 0, \pm 1, \pm 2, \cdots$. (In higher dimensions, one would have to consider the usual Laplacian $\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$.) Thus the Fourier inversion (expansion) formula as well as the Fourier-series formula can be thought of as an eigenfunction expansion for the Laplacian.

Suppose f represents the amplitude of a signal (say a sound wave or a light wave) at time t. Then the formulae:

$$\hat{f}(w) = \int_{-\infty}^{\infty} f(t)e^{-iwt}dt \text{ and } f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwt}dw$$

really express the fact that f is a continuous superposition of the simple periodic waves e^{iwt}, as w ranges over all possible frequencies. For instance, if one already knows that the dominating frequencies making up f are mostly concentrated around w_0, say in an interval $[w_0 - a, w_0 + a]$, then this is reflected in the mathematical fact that $\hat{f}$ is mostly concentrated in the interval $[w_0 - a, w_0 + a]$. In this case, a reasonable approximation for f would be $f(t) \overset{\text{approx}}{=} \frac{1}{2\pi}\int_{w_0-a}^{w_0+a} \hat{f}(\lambda)e^{i\lambda x}d\lambda$. If we know that *no* frequency outside the interval $[-\Omega, \Omega]$ is involved, i.e. $\hat{f}$ vanishes outside $[-\Omega, \Omega]$, then a remarkable theorem of Shannon's (the celebrated information theorist), the so-called *Shannon sampling theorem*, says that f can be completely recovered from its values at the points $t_n = \frac{n\pi}{\Omega}, n = 0, \pm 1, \pm 2, \cdots$ via the formula $f(t) = \sum_{n=-\infty}^{\infty} f(\frac{n\pi}{\Omega})\frac{\sin(\Omega t - n\pi)}{(\Omega t - n\pi)}$. (Of course, f has to be reasonably well behaved and square integrable.) For both a heuristic as well as a rigorous justification of this fact, see pages 230-231 of [3].

Finally, we urge the reader to look up the books [2] and [3] for details regarding the theory of Fourier series and Fourier transforms and also the many important applications of this theory to other branches of mathematics, physics, electrical engineering etc.

Suggested Reading

[1] S Thangavelu. Fourier Series. *Resonance.* Vol 1,No 10,1996.

[2] T W Körner. *Fourier Analysis.* Cambridge University Press, U.K., 1988.

[3] G B Folland. *Fourier Analysis and its Applications.* Wadsworth and Brooks/Cole, U.S.A., 1992.

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