From Fourier Series to Fourier Transforms

Thanks to the pioneering work of Bernoulli, Euler, Lagrange and Fourier, today every student of science and engineering knows that a periodic function $f(x)$ of period $2\pi$ can be expanded (possibly in an infinite series) in terms of $\sin kx$ and $\cos kx, k = 0, 1, 2, \cdots$ (see [1].) For more on the historical developments that led to this idea, the reader is encouraged to consult [2]. Thus, using complex notation, a $2\pi$-periodic function $f(x)$ on the real line $\mathbb{R}$ can be ‘written’ in the form

$$f(x) \approx \sum_{n=-\infty}^{\infty} a_n e^{inx} \quad (\star)$$

where, as Fourier observed, the $a_n$'s are given by the formula $a_n = \frac{1}{2\pi} \int_{0}^{2\pi} f(x)e^{-inx}dx$. In what sense does the series on the right converge, and if it does converge, in what sense is it equal to $f(x)$? These are surprisingly subtle matters and the answers depend on the nature of the function $f(x)$. For instance, if, as is the case in most applications, $f$ is square-integrable on $[0,2\pi]$, i.e. $\int_{0}^{2\pi} |f(x)|^2dx < \infty$, then the series on the right hand side of $(\star)$ is absolutely square summable on $[0,2\pi]$, i.e. $\sum_{n=-\infty}^{\infty} |a_n|^2 < \infty$, and converges in the mean square sense to $f$ i.e. $\int_{0}^{2\pi} |f(x) - \sum_{k=-n}^{n} a_ke^{ikx}|^2dx \to 0$ as $n \to \infty$. If $f$ is reasonably smooth, then the series converges uniformly to $f$. For more on this fascinating story the reader is referred to [2].

The appearance of the function $e^{inx}$ is not entirely accidental! Note that the interval $[0,2\pi]$, with the end points identified, can be thought of as the unit circle $S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$, via the identification $\theta \longleftrightarrow e^{i\theta}$. Under this identification, the function $e^{inx}$ is really the function $\chi_n$ on $S^1$ defined by $\chi_n(z) = z^n$. $S^1$ is a group under multiplication and the $\chi_n$, $s, n = 0, \pm 1, \pm 2, \cdots$ account for all the continuous homomorphisms of $S^1$ into $S^1$. We now pass on to the case of functions $f(x)$ defined on $\mathbb{R}$ which satisfy a reasonable condition like $\int_{-\infty}^{\infty} |f(x)|dx < \infty$ (integrability) or $\int_{-\infty}^{\infty} |f(x)|^2dx < \infty$ (square integrability). Notice that such functions (which are not identically zero) cannot be periodic.
(Why not?) So, no matter how we re-scale the real line, we cannot really hope to ‘expand’ such functions in terms of \( \sin kx \) and \( \cos kx, k = 0, 1, 2, \cdots \), as in the periodic case. Taking the cue from the fact that the functions \( e^{inx} \) correspond to the homomorphisms \( \chi_n \) of \( S^1 \) into \( S^1 \), we look for continuous homomorphisms of \( \mathbb{R} \) into \( S^1 \). It turns out that all such continuous homomorphisms of \( \mathbb{R} \) into \( S^1 \) are given by \( \varphi_{\lambda}(x) = e^{i\lambda x}, \lambda \in \mathbb{R} \). (\( \mathbb{R} \) is a group with + as the group law.) So we are tempted to conjecture that \( f(x) \) can be ‘expanded’ in terms of \( e^{i\lambda x} \). Clearly since we have a continuum of such functions, we may have to replace the infinite series in the periodic case by an integral in the non-periodic case. That this can be done is one of the great triumphs of Fourier transform theory. To cut a long story short, for an integrable function \( f(x) \), define \( \hat{f}(\lambda) \), the Fourier transform of \( f(x) \), by
\[
\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx.
\]
Then it turns out that \( \hat{f}(\lambda) \) is a continuous function on \( \mathbb{R} \), and \( |\hat{f}(\lambda)| \to 0 \) as \( |\lambda| \to \infty \). If the function \( f \) is sufficiently well behaved, then we can recover \( f(x) \) from \( \hat{f}(\lambda) \) by the Fourier inversion formula \( f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\lambda x} d\lambda \). Moreover if \( f \) is also square-integrable, then we do have mean square convergence as in the Fourier series case, i.e.
\[
\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 d\lambda.
\]
Actually there is a purely heuristic way of justifying the Fourier inversion formula starting from the Fourier expansion of a periodic function. (See pages 204-205 of [3]).

There is another, perhaps intuitively more appealing way of motivating the Fourier transform. Notice that the Fourier inversion formula is an expansion of \( f \) in terms of the functions \( \varphi_{\lambda}(x) = e^{i\lambda x} \). Now each \( \varphi_{\lambda} \) is an eigenfunction for the one-dimensional Laplace operator \( \frac{d^2}{dx^2} \) with eigenvalue \(-\lambda^2\). Among these the \( 2\pi \)-periodic ones are precisely \( e^{ikx}, k = 0, \pm 1, \pm 2, \cdots \). (In higher dimensions, one would have to consider the usual Laplacian \( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \).) Thus the Fourier inversion (expansion) formula as well as the Fourier-series formula can be thought of as an eigenfunction expansion for the Laplacian.

Suppose \( f \) represents the amplitude of a signal (say a sound wave or a light wave) at time \( t \). Then the formulae:
\[ \hat{f}(w) = \int_{-\infty}^{\infty} f(t) e^{-iwt} dt \quad \text{and} \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwt} dw \]

really express the fact that \( f \) is a continuous superposition of the simple periodic waves \( e^{iwt} \), as \( w \) ranges over all possible frequencies. For instance, if one already knows that the dominating frequencies making up \( f \) are mostly concentrated around \( w_0 \), say in an interval \([w_0 - a, w_0 + a]\), then this is reflected in the mathematical fact that \( \hat{f} \) is mostly concentrated in the interval \([w_0 - a, w_0 + a]\). In this case, a reasonable approximation for \( f \) would be \( f(t) \approx \frac{1}{2\pi} \int_{w_0-a}^{w_0+a} \hat{f}(\lambda) e^{i\lambda t} d\lambda \).

If we know that no frequency outside the interval \([-\Omega, \Omega]\) is involved, i.e. \( \hat{f} \) vanishes outside \([-\Omega, \Omega]\), then a remarkable theorem of Shannon’s (the celebrated information theorist), the so-called Shannon sampling theorem, says that \( f \) can be completely recovered from its values at the points \( t_n = \frac{n\pi}{\Omega}, n = 0, \pm1, \pm2, \ldots \) via the formula \( f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{\Omega}\right) \frac{\sin(\Omega t - n\pi)}{(\Omega t - n\pi)} \). (Of course, \( f \) has to be reasonably well behaved and square integrable.) For both a heuristic as well as a rigorous justification of this fact, see pages 230-231 of [3].

Finally, we urge the reader to look up the books [2] and [3] for details regarding the theory of Fourier series and Fourier transforms and also the many important applications of this theory to other branches of mathematics, physics, electrical engineering etc.

**Suggested Reading**

