Reflections

'A Mathematician's Apology' by the famous British mathematician G H Hardy was reviewed in *Resonance* (December 1996). Generally speaking Hardy's book was given an enthusiastic and uncritical reception. The book is such a beautiful piece of prose and written so convincingly, that not many questioned Hardy's point of view. L J Mordell himself a distinguished mathematician, dared to question Hardy's point of view in a bold review of Hardy's classic which is reproduced here with permission.

Hardy's " A Mathematician's Apology"

L J Mordell

A reprint of this most interesting book appeared in 1967 with a foreword by C.P. Snow, Hardy's friend of long standing.

It has often been reviewed and highly praised, but there are, however, some opinions expressed by Hardy which, perhaps, have not been adequately dealt with by other reviewers. Furthermore, Snow's foreword calls for some comment, especially his references to Ramanujan (1887–1920). He writes that after Hardy and Littlewood read the manuscript sent by Ramanujan to Hardy (probably Jan. 16, 1913), "they knew, and knew for certain" that he "was a man of genius .... It was only later that Hardy decided that Ramanujan was, in terms of natural mathematical genius, in the class of Gauss and Euler; but that he could not expect, because of the defects of his education, and because he had come on the scene too late in the line of mathematical history, to make a contribution on the same scale." While one would readily accept that Ramanujan was a man of genius, the comparison with Gauss and Euler is very farfetched. I have some difficulty in believing that Hardy made such a statement, or at any rate made it in this form. What does natural mathematical genius mean? Undoubtedly Ramanujan was outstanding in some aspects of mathematics and had great potentialities. But this is not enough. What really matters is what he did, and

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one cannot accept such a comparison with Euler and Gauss, whose many-sided contributions were of fundamental importance and changed the face of mathematics. In fact in 1940, in the book on Ramanujan, Hardy said "I cannot imagine anybody saying with any confidence, even now, just how great a mathematician he was and still less how great a mathematician he might have been."

Snow says that Ramanujan, as is commonly believed, was the first Indian to be elected (2 May, 1918) a fellow of the Royal Society. He was the second. The first was Ardaseer Cursetjee (1808–1877), shipbuilder and engineer, F.R.S. 27 May, 1841. Snow notes that Ramanujan was elected a fellow of Trinity four years after his arrival in England and continues, "it was a triumph of academic uprightness that they should have elected Hardy's protégé Ramanujan at a time when Hardy was only just on speaking terms with some of the electors and not at all with others." It is well that the merits of a fellowship candidate are judged by the quality of his original work and not by the political views of his sponsors.

Let us examine some of the views expressed by Hardy. They are sometimes stated too categorically, regardless of exceptions and limitations. A number of them had their origin in what he says, most gloomily, in the very first section of Apology: "It is a melancholy experience for a professional mathematician to find himself writing about mathematics. The function of a mathematician is to do something, to prove new theorems, to add to mathematics, and not to talk about what he or other mathematicians have done."

His practice many years ago does not conform with this statement. He recalls in Section 6 that he did talk about mathematics in his 1920 Oxford inaugural lecture, which actually contains an apology for mathematics. Further in 1921, he gave an address on Goldbach's theorem to the Mathematical Society of Copenhagen. In this, he did talk about what he and other mathematicians had done. Such talks render a real service to mathematics and many have found great pleasure and inspiration in listening to or reading such expositions. Hardy had followed the practice of many eminent mathematicians in giving them. These have contributed to the richness and vividness of mathematics and make it a living entity. Without them, mathematics would be much the poorer.

No mathematician can always be producing new results. There must inevitably be fallow periods during which he may study and perhaps gather ideas and energy for new work. In the interval, there is no reason why he should not occupy himself with various aspects of mathematical activity, and every reason why he should. The real function of a mathematician is the advancement of mathematics. Undoubtedly, the production of new results is the most important thing he can do, but there are many other activities which he can initiate or participate in. Hardy had his full share of these. He took a leading part in the reform of the mathematical tripos some sixty years ago. Before then, it was looked upon as a sporting event, reminding one of the Derby, and was out of touch with continental mathematics. A mathematician can engage in the many administrative aspects of mathematics. Hardy was twice Secretary and President of the London Mathematical Society and, while so occupied, must have done an enormous amount of unproductive work. He served on many committees dealing with mathematics and mathematicians. He wrote a great many obituary notices. He was well aware that a professor of mathematics is a representative of his subject in his University. This entails many duties which cannot be called doing mathematics.

His reference to a melancholy experience shows how much he took to heart and suffered from the loss of his creative powers. The result is, as Snow says, that the *Apology* is a book of haunting sadness.

Further in this first section, he says despairingly, "if then I find myself writing not mathematics but 'about' mathematics, it is a confession of weakness, for which I may rightly be scorned or pitied by younger and more vigorous mathematicians. I write about mathematics because, like any other mathematician who has passed sixty, I have no longer the freshness of mind, the energy, or the patience to carry on effectively with my proper job." He had been for many years a most active mathematician and his collected works now being published will consist of seven volumes. It seems almost nonsense to say that anyone would scorn or pity him, and the use of the term 'rightly' is even more nonsensical.

We all know only too well that with advancing age we are no longer in our prime, and that our powers are dimmed and are not what they once were. Most of us, but not Hardy, accept the inevitable. There are still many consolations. We can perhaps find pleasure in thinking about some of our past work. We can read what others are doing, but this may not be easy since many new techniques have been evolved, sometimes completely changing the exposition of classical mathematics. Various reviews, however, may give one some idea what has been done. (We can still be of service to younger mathematicians.)

His statement about a mathematician who has passed sixty is far too sweeping and any number of instances to the contrary can be mentioned, even among much older people. One need only note some recent Cambridge and Oxford professors.

Great activity among octogenarians is shown by Littlewood, his lifelong collaborator, Sydney Chapman, his former pupil and collaborator, and myself. There is also Besicovitch in the seventies. Davenport, who had passed sixty, was as active and creative as ever, and his recent death is a very great loss to mathematics since he could have been expected to continue to produce beautiful and important work.

The question of age was ever present in Hardy's mind. In Section 4, he says, "No mathematician should ever allow himself to forget that mathematics, more than any other art or science, is a young man's game." It seems that he could not reconcile himself to growing old. For further on, he says, "I do not know an instance of a major mathematical advance initiated by a man past fifty". This may be so, but much depends on the definition of the *advance*. But there is no need to be troubled about it. Much important work has been done by men after the age of fifty.

A number of Hardy's statements must be qualified. In Section 2, he says that "good work is not done by 'humble' men. It is one of the first duties of a professor, for example, in any subject, to exaggerate a little both the importance of his subject and his own importance in it. A man who is always asking, 'Is what I do worthwhile?' and 'Am I the right person to do it?' will always be ineffective himself and a discouragement to others."

Though one may naturally have a better opinion of one's work than others have, there are many exceptions to his statement. I never knew Davenport to exaggerate or emphasize the importance of his work, but he was a most effective mathematician and a very successful supervisor of research. Prof. Frechet told me a few years ago, that when Norbert Wiener was working with him a long time ago, he was always asking, "Is my work worthwhile?", "Am I slipping?", etc. S Chowla is as modest and humble a mathematician as I know of, but he inspires many research students.

We comment on some more of Hardy's statements about mathematics. One of the most surprising is in Section 29, "I do not remember having felt, as a boy, any *passion* for mathematics, and such notions as I may have had of the career of a mathematician were far from noble. I thought of mathematics in terms of examinations and scholarships; I wanted to beat other boys, and this seemed to me to be the way in which I could do so most decisively."

It has often been said that mathematicians are born and not made. Most great mathematicians developed their keenness for mathematics in their school days. Their ability revealed itself by comparison with the performances of their schoolmates. Their ambition was to continue the study of mathematics and to take up a mathematical career. Probably no other motive played any part in the decision of most of them.

In Section 3, he considers the case of a man who sets out to justify his existence and his activities. I see no need for justification any more than a poet or painter or sculptor does. As Trevelyan says, disinterested intellectual curiosity is the life blood of real civilization. It is curiosity that makes a mathematician tick. When Fourier reproached Jacobi for trifling with pure mathematics, Jacobi replied that a scientist of Fourier's calibre should know that the end of mathematics is the great glory of the human mind. Most mathematicians do mathematics for the good reason that they like and enjoy doing it. Davenport told me that he found it "exciting" to do mathematics.

Hardy says that the justifier has to distinguish two different questions. The first is whether the work which he does is worth doing and the second is why he does it, whatever its value may be. He says to the first question: The answer of most people, if they are honest, will usually take one or other of two forms; and the second form is merely a humbler version of the first, which we need to consider seriously. "I do what I do because it is the one and only thing I can do at all well." It suffices to say that the mathematician felt no need to do anything else.

Hardy is appreciative of the beauty and aesthetic appeal of mathematics. "A mathematician," he says in Section 10, "like a painter or poet, is a maker of patterns ... ," and these "... must be beautiful. The ideas ... must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics .... It may be very hard to *define* mathematical beauty..." but one can recognize it. He discusses the aesthetic appeal of theorems by Pythagoras on the irrationality of $\sqrt{2}$ and Euclid on the existence of an infinity of prime numbers. He says in Section 18, "there is a very high degree of *unexpectedness*, combined with *inevitability* and *economy*. The arguments take so odd and surprising a form; the weapons used seem so childishly simple when compared with the far-reaching results."

I might suggest among other attributes of beauty, first of all, simplicity of enunciation. The meaning of the result and its significance should be grasped immediately by the reader, and these in themselves may make one think what a pretty result this is. It is, however, the proof which counts. This should preferably be short, involve little detail and a minimum of calculations. It leaves the reader impressed with a sense of elegance and wondering how it is possible that so much can be done with so little.

Somehow, I do not think that Hardy's work is characterized by beauty. It is distinguished more by his insight, his generality, and the power he displays in carrying out his ideas. Many of the results that he obtains are very important indeed, but the proofs are often long and require concentrated attention, and this may blunt one's feelings even if the ideas are beautiful.

Hardy does not define ugly mathematics. Among such, I would mention those involving considerable calculations to produce results of no particular interest or importance; those involving such a multiplicity of variables, constants and indices, upper, lower, right and left, making it very difficult to gather the import of the result; and undue generalization apparently for its own sake and producing results with little novelty. I might also mention work which places a heavy burden on the reader in the way of comprehension and verification unless the results are of great importance.

Hardy had previously said that he could "quote any number of fine theorems from the theory of numbers whose meaning anyone can understand, but whose proofs, though not difficult, may be found tedious." It often happens that there are significant results apparently of some depth, the proof of which can be grasped by those with a minimum of mathematical knowledge. Perhaps I may be pardoned if I give one of my own. The theorem of Pythagoras suggests the problem of finding the integer solutions of the equation $x^2+y^2=z^2$. This was done some 1000 years ago and is not difficult. But suppose we consider the more general equation $ax^2 + by^2 = cz^2$. This is a real problem in the theory of numbers. Legendre at the end of the eighteenth century gave necessary, and sufficient conditions for its solvability. Then when the equation is taken in the normal form, i.e. abc is square-free and $a > 0, b > 0, c > 0$, Holzer showed in 1953 that a solution existed with $|z| < \sqrt{ab}$, from which it follows that $|x| \leq \sqrt{bc}$, $|y| \leq \sqrt{ca}$. I recently, found a proof of this result that no one would call tedious by showing that if a solution (x_1,y_1,z_1) existed with $|z_1| > \sqrt{ab}$, then there was another with $|z_2| < |z_1|$. This arose by taking an appropriate line through the point (x_1,y_1,z_1) to meet the conic $ax^2+by^2=cz^2$ in the point (x_2,y_2,z_2). I call this a schoolboy proof, because the only advanced result required is that the equation $lx + my = n$ has an integer solution if l and m are co-prime. A proof of the theorem could have been found by a schoolboy.

We conclude by examining Hardy's views about the utility or usefulness of mathematics. He seems to denigrate the usefulness of 'real' mathematics. In Section 21, he says, "The 'real' mathematics of the 'real' mathematicians, the mathematics of Fermat and Euler and Gauss and Abel and Riemann is almost wholly 'useless'." This statement is easily refuted.

A ton of ore contains an almost infinitesimal amount of gold, yet its extraction proves worthwhile. So if only a microscopic part of pure mathematics proves useful, its production would be justified. Any number of instances of this come to mind, starting with the investigation of the properties of the conic sections by the Greeks and their application many years later to the orbits of the planets. Gauss' investigations in number theory led him to the study of complex numbers. This is the beginning of abstract algebra, which has proved so useful for theoretical physics and applied mathematics. Riemann's work on differential geometry proved of invaluable service to Einstein for his relativity theory. Fourier's work on Fourier series has been most useful in physical investigations. Finally one of the most useful and striking applications of pure mathematics is to wireless telegraphy which had its origin in Maxwell's solution of a differential equation. Many new disciplines are making use of more and more pure mathematics, e.g., the biological sciences, economics, game theory and communication theory, which requires the solution of some difficult Diophantine equations. It has been truly said that advances in science are most rapid when their problems are expressed in mathematical form. These in time may lead to advances in pure mathematics.

These remarks may serve as a reply to Hardy's statement that the great bulk of higher mathematics is useless.

It is suggested that one purpose mathematics may serve in war is that a mathematician may find in mathematics an incomparable anodyne. Bertrand Russell says that in mathematics, "one at least of our nobler impulses can best escape from the dreary exile of the actual world." Hardy's comment on this reveals his depressed spirits. "It is a pity," he says, "that it should be necessary to make one very serious reservation – he must not be too old. Mathematics is not a contemplative but a creative subject; no one can draw much consolation from it when he has lost the power or desire to create; and that is apt to happen to mathematicians rather soon." What does he mean when he says mathematics is not a contemplative subject? Many people can derive a great deal of pleasure from the contemplation of mathematics, e.g., from the beauty of its proofs, the importance of its results, and the history of its development. But alas, apparently not Hardy.

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