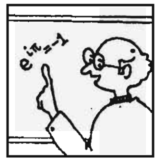

In this section of Resonance, *we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.*

Buffon's Needle Problem Revisited

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One of the oldest problems involving ideas of 'geometric probability' (as opposed to combinatorial probability), the Buffon's Needle Problem was stated in the May 1997 issue of *Resonance* and its solution appeared in the October 1997 issue. It was first posed by Georges L L Comte de Buffon (1707–1788), a French naturalist, in the *Proceedings of the Paris Academy of Sciences,* 1733 and solved by him in his book *Essai d'arithmetique morale* published in 1777. The problem is to find the probability that a needle (of length l) thrown at random on a board ruled with a set of equidistant parallel lines b units apart with ($b > l$) will intersect one of the lines. This probability turns out to be $2\,l\,/\pi b$. Later, Pierre Simon de Laplace (1749–1827) remarked that this can be used to find an approximation to π since the probability is approximately the proportion of times the needle intersects one of the lines when the experiment is repeated a large number of times. Results of such experiments done in the previous centuries may be found in the reference cited at the end, where a different and ingenious solution to Buffon's needle problem due to Barbier can also be found.

Here we consider a generalisation of Buffon's needle problem due to Laplace and give its solution.

Laplace's Problem

A board is covered with a set of identical rectangles and a thin needle is thrown on the board. Assuming that the needle is shorter than the smaller side of the rectangles, what is the probability that the needle will be entirely contained in one of the rectangles of the set?

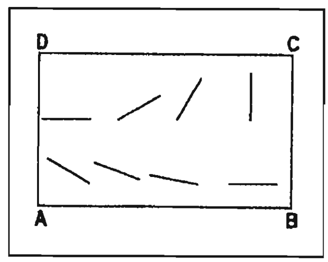
Solution

First, assume that the board consists of only one rectangle of length a and breadth b $(b < a)$. It is reasonable to say that the needle of length $l < b$ falls (and stays) on the board if and only if the midpoint of the needle is in the board – otherwise the needle would fall to the ground and such cases are not relevant to the problem. The event E whose probability is required is that the needle lies entirely within the rectangle.

Let the rectangle be $ABCD$ with $AB=a, BC=b$, and take AB as the x axis and AD as the y axis with A as the origin (see *Figure 1*). Considering various positions of the needle, we can describe any position uniquely by the coordinates (x, y) of its midpoint and the angle θ that the needle makes with the x axis where :

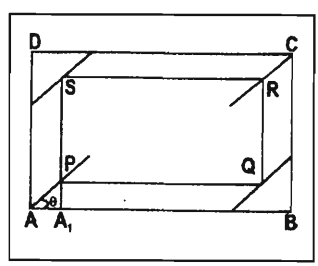
$$0 < x < a, 0 < y < b, 0 < \theta < \pi \ (*).$$

Figure 1.



As the needle is thrown at random, it is reasonable to assume that each of these variables has uniform distribution in its domain of variation and these are independently distributed. Thus for the joint distribution of x, y and θ, we have uniform distribution in the cuboid (*) above. To find the probability of the event E, we need to evaluate the volume of the region V inside (*)where the condition for the event E will hold. We will find this volume by first fixing θ and finding the area of the θ-section of V and then integrating over θ.

Figure 2.



So let θ be fixed and let $0 < \theta \leq \pi/2$. A moment's reflection will convince us (see *Figure 2*) that the event E occurs if and only if the midpoint (x,y) of the needle lies inside the rectangle $PQRS$

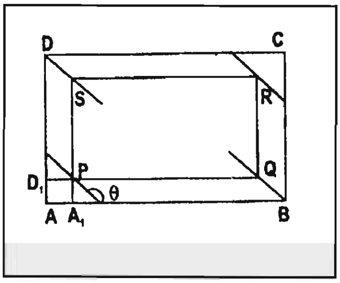
where $\angle PAB = \theta, AP = l/2$ and the other three slant lines (extreme positions of the needle such that it lies within the rectangle) are parallel to AP. Let SP (produced) meet AB at A_1.

Then

$$PQ = AB - 2AA_1 = a - l\cos\theta \text{ and}$$
$$PS = AD - 2A_1P = b - l\sin\theta.$$

Hence the area of rectangle $PQRS$ in this case is $(a - l\cos\theta)(b - l\sin\theta)$.

Figure 3.



Now let $\pi/2 < \theta < \pi$. In this case also (see *Figure 3*) we get a rectangle $PQRS$ such that the needle will be entirely within $ABCD$ if and only if its midpoint falls within $PQRS$. Produce SP to meet AB at A_1 and produce QP to meet AD at D_1. Then $AD_1 = l/2 \sin(\pi - \theta) = l/2 \sin\theta$ and $AA_1 = l/2 \cos(\pi - \theta) = l/2\,|\cos\theta|$. The required area in this case is therefore $(a - l\,|\cos\theta|)(b - l\sin\theta)$.

The volume of the favourable region V is therefore:

$$\int_0^{\pi/2} (a - l\cos\theta)\,(b - l\sin\theta)\,\mathrm{d}\theta + \int_{\pi/2}^{\pi} (a - l\,|\cos\theta|)(b - l\sin\theta)\,\mathrm{d}\theta$$

which can be evaluated as $\pi ab - 2l(a+b) + l^2$. The total volume of the cuboid (*) is πab and so the required probability p is $1 - (2l(a+b) - l^2)/\pi ab$.

Consider now the general case when the board consists of mn rectangles (each having length a and breadth b) such that there are m divisions on the longer side and n divisions on the shorter side. By what we said in the beginning, it is reasonable to say that the needle has fallen on the k^{th} rectangle, R_k say, if its midpoint lies inside R_k. (The event that the midpoint of the needle falls on the boundary of R_k has zero probability in this set up.) Let E_k denote the event that the needle lies entirely inside R_k and F_k the event that the midpoint of the needle lies inside R_k. Note that the F_k's are disjoint events and so also are the E_k's. Then the probability p found above is the conditional probability of the event E_k given that F_k has occurred. It is now clear that the probability that the needle falls entirely inside one of the rectangles is given by:

$$\sum_k P(E_k) = \sum_k P(E_k \mid F_k) P(F_k) = mn \cdot p \cdot \frac{1}{mn} = p$$

as there are a total of mn rectangles. The probability that the needle will intersect one of the rectangles is therefore $1 - p = (2\,l\,(a+b) - l^2)\,/\pi ab$.

Buffon's needle problem can be considered as a limiting case when $a \to \infty$; b being the distance between a set of parallel lines on the plane. The probability that the needle will intersect one of the lines is therefore

$$\lim_{a \to \infty} \frac{2l(a+b) - l^2}{\pi\, ab} = \frac{2l}{\pi b}.$$

We now consider a related problem whose solution however uses that of Buffon's needle problem.

Plate Problem

A thin plate in the shape of a convex polygon, of dimensions so small that it cannot intersect two of the lines simultaneously, is thrown on a board ruled as in Buffon's needle problem. What is the probability that the boundary of the plate will intersect one of the lines?

Note that a line can intersect either none or exactly two sides of the plate. The answer turns out to be the ratio of the perimeter of the plate to πb, where b is the distance between the parallel lines. (For the solution, see Suggested Reading). One may think of a line as a polygon with two sides and then its perimeter is indeed twice its length! So we are back at Buffon's needle problem.

Other significant contributors to the general area of geometric probability are W Crofton, E Czuber, Sylvester and H Poincaré.

Suggested Reading

J V Uspensky. *Introduction to Mathematical Probability.* McGraw-Hill Book Company, Inc. Reprinted in India by Tata McGraw-Hill Publishing Co. pp. 112–115 and pp. 251–256, 1937.