Teaching The Limit Concept

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"Once more into the breach, dear friends, once more"
Shakespeare, King Henry V, III.1.1

It is the experience of all mathematics teachers that at first contact students face a lot of difficulty with formal analysis. The difficulties are two fold. As Tall says in [1], "First the student usually imagines the definition to *describe* an existing object, rather than *define* the object by deducing its properties, thus finding it strange to 'prove' obvious properties that seem already to be true. Then there are further difficulties because of the *complex use of quantifiers* [emphasis added] and the formality of the deductions. The decision of most UK universities to abandon the teaching of formal analysis as a first year university course is evidence of its huge cognitive difficulty."

Difficult or not, at some stage we have to tackle the problem of teaching formal analysis. The question then is how best we should go about it. At the heart of analysis is the concept of limit, so let us consider how the limit concept could be taught.

To start with we need to decide whether it is possible, and if so whether it is advisable, to present the limit concept in a way which avoids the standard ε–δ formulation. That this is possible has been shown by Hijab in [2]. His approach is the following. Define the limit of a monotone sequence as a supremum or infimum as the case may be. Then define the limit superior and limit inferior of a sequence as limits of appropriate monotone subsequences. When they are equal define the limit of the sequence to be their common value. Finally, define continuity in terms of sequences. The back cover of [2] stresses the fact that ε's and δ's have been done away with and uses it to advertise the book.

There are other interesting innovations in the book and one may want to consider using it as a text for some special batches of students. However in the general Indian context it may not be wise to depart too radically from the traditional approach (say that of Hardy; see [3]). As our students have to seek admission in universities and in the process have to face examiners with varying backgrounds, it may be safer to keep as close as possible to the classical approach.

In [4], Berberian suggests ways of softening the impact of the quantifiers by introducing stepping-stone concepts such as 'ultimately', 'frequently' and 'null sequence'. A sequence is said to be *ultimately* in a set S if all its entries from some point on are in S. It is called *null* if it is ultimately in every set of the form $(-\varepsilon, \varepsilon)$. We then say that a sequence (s_n) converges to s if $(|s_n-s|)$ is a null sequence. A sequence is *frequently* in a set S if infinitely many entries of the sequence are in S. This concept is useful in talking about limit points and in studying the properties of limsup and liminf of a sequence. The approach seems promising, but the student still has to come to grips with quantifiers in order to understand the proofs.

The purpose of this article is to suggest a way of avoiding the use of quantifiers by taking a cue from the subjects of predicate logic and automated reasoning, where use is made of a technical device known as Skolemisation[1] which reduces complex propositions involving quantifiers to a standard form without quantifiers, so that it is in principle easy for a computer to check them. In terms of pedagogy this approach seems a promising one. We briefly present the idea below with some heuristic motivation.

[1] Thoralf A Skolem was a mathematician who contributed to logic, set theory and algebra in the early twentieth century. Suppes in [5] says that what is now known as Zermelo–Fraenkel set theory should in all fairness be called Zermelo–Fraenkel–Skolem set theory.

Let $f : \boldsymbol{R} \rightarrow \boldsymbol{R}$ be a function. We think of f as an 'input-output box', with an input of a giving rise to an output of $f(a)$. Now imagine that there is an error in the input a; say we input instead a number x which is close to a. The output would then be $f(x)$ and the error in the output would be $|f(x) - f(a)|$. So an input error of $|x - a|$ has led to an output error of $|f(x) - f(a)|$. It is clearly desirable that we are able to control the error in the output by controlling the error in the input.

Definition. A *control function* for f at a point a is a function $\delta : (0, \infty) \to (0, \infty)$ such that for all $\varepsilon > 0$ the following holds:

If $|x - a| < \delta(\varepsilon)$,
then $|f(x) - f(a)| < \varepsilon$.

If f has a control function at a then f is said to be *continuous* at a.

At first glance the definition may seem to be merely a rephrasing of the standard $\varepsilon - \delta$ definition. However some reflection and a little experimentation show that this apparently minor change allows us to present the standard proofs in a logically simpler form. To illustrate this, we present a standard proposition and indicate its proof.

Proposition. Suppose that f and g are continuous at a. Then (1) $f + g$ and (2) $f \cdot g$ are continuous at a.

Proof. Let δ_1 and δ_2 be control functions at a for f and g respectively.
(1) Let $\delta(\varepsilon) = \min\{\delta_1(\varepsilon/2), \delta_2(\varepsilon/2)\}$ for all $\varepsilon > 0$ Then δ is a control function for $f + g$ at a.
(2) Let $M_1 = |f(a)| + 1, M_2 = |g(a)| + 1$. Define the function δ on $(0, \infty)$ thus:

$$\delta(\varepsilon) = \min\left\{\delta_1\left(\frac{\varepsilon}{2M_2}\right), \delta_2\left(\frac{\varepsilon}{2M_1}\right), \delta_1(1)\right\}.$$

Now suppose that $|x - a| < \delta(\varepsilon)$. Then $|x - a| \le \delta_1(1)$, so $|f(x)| < |f(a)| + 1 = M_1$. Therefore

$$|f(x)g(x) - f(a)g(a)| \le |f(x)||g(x) - g(a)| + |g(a)||f(x) - f(a)|$$

$$< M_1 \cdot \frac{\varepsilon}{2M_1} + M_2 \cdot \frac{\varepsilon}{2M_2} = \varepsilon,$$

so the implication '$|x - a| < \delta(\varepsilon) \Rightarrow |f(x)g(x) - f(a)g(a)| < \varepsilon$' holds. Thus δ is a control function for $f \cdot g$ at a.

It is a simple exercise to write the proofs of various propositions

Suggested Reading

[1] Tall D O. Functions and Calculus. in *International Handbook of Mathematical Education Part 1.* Alan J Bishop et al (eds) Kluwer Acad. Pub., Dordrecht, 1996.

[2] Hijab O. *Introduction to Calculus and Classical Analysis.* Springer-Verlag Undergraduate Texts in Mathematics, 1997.

[3] Hardy G H. *A Course of Pure Mathematics.* Cambridge Univ. Press, 1967 (10th edition).

[4] Berberian S K. *A First Course in Real Analysis.* Springer-Verlag, 1994.

[5] Suppes P. *Axiomatic Set Theory.* Van Nostrand, 1960.

[6] Bledsoe W W. Some Automatic Proofs in Analysis. in *Automated Theorem Proving: After 25 Years.* (eds) W W Bledsoe and D W Loveland, Contemporary Mathematics. 29. *Am. Math. Soc.*, 1984.

concerning limits and continuity in terms of control functions. Note that one can similarly define the limit of a sequence in terms of a control function from $(0, \infty)$ to the set of natural numbers $\boldsymbol{N}$.

A case can readily be made in favour of using control functions to teach the limit concept, but whether it actually 'works' (i.e., whether students really benefit) can only be decided after some experimentation by a body of teachers. It is hoped that this article will provide the necessary stimulus for such experimentation.

Please Note

In 'Special Relativity – An Exoteric Narrative: Wherein we put formulas in their place', *Resonance* Classroom Section, Vol.3, No.5, pages 63–72 the figures should appear as follows:

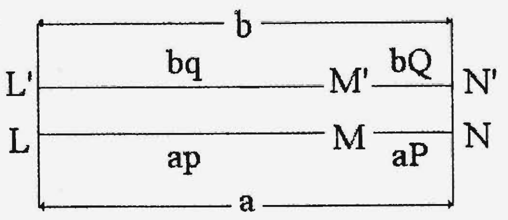

Figure 1.

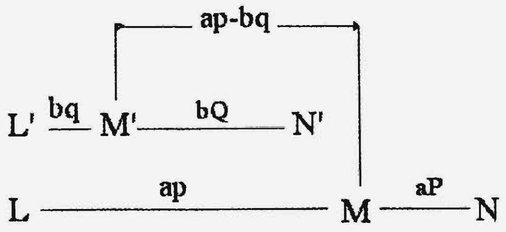

Figure 2.

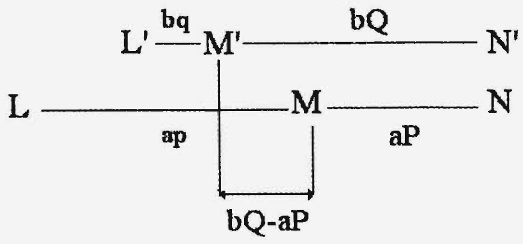

Figure 3.

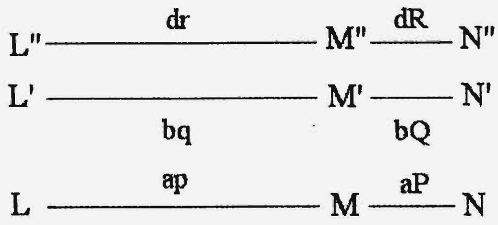

Figure 4.