Chaos

2. Routes to Chaos

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[1] Part 1 of this series 'Introduction to chaos' appeared in *Resonance*, Vol.3, No.4, 1998.

In the first part [1] we described chaos and emphasised that non-linearity is an important ingredient of it. In this part we discuss the different routes that are taken by a system before it becomes chaotic.

Routes to Chaos

Given a dynamical system, the basic feature one would like to know is whether the motion is chaotic or periodic, and how this changes as a function of system parameters, as we have seen happening as a function of E the driving amplitude in the damped oscillator.

These questions are often easier to answer in the context of *maps* which are discrete dynamical systems, and which can sometimes be related to continuous dynamical systems (described by differential equations) as follows. Given a trajectory in a N-dimensional phase space, if one were to place a $N-1$ dimensional surface in the phase space, the trajectory would typically (unless the trajectory was peculiar and the $N-1$ dimensional surface was chosen very atypically!) intersect with this surface at points $(x_{1n}, x_{2n}, \ldots, x_{(N-1)n})$, with n now denoting the nth intersection. Such a construction is known as the Poincaré surface of section (*Figure 1*). The time evolution of the differential equations is

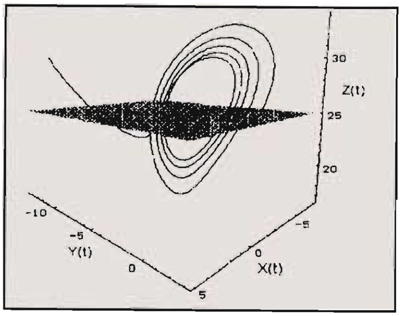


Figure 1. Poincaré section of a trajectory in 3-D phase space by a plane.

mimicked by the following iterative system,

$$x_{i\ n+1} = M_i(x_{1n}, x_{2n}, \dots x_{(N-1)n}) \qquad (1)$$

where M is the mapping implicitly defined by the construction above. Not all maps need be thought of as the Poincaré sections of higher dimensional systems, and in most cases, they cannot be (see below).

It is simpler to study the mapping: numerically, this is through simple iteration, while the study of the differential equations

Poincaré Section

It is fairly easy to imagine the phase space trajectory of a one-dimensional damped simple harmonic oscillator (or a simple pendulum with friction), for which the phase space is two-dimensional (see *Figure 6* in part 1). But it is much harder to visualise the trajectory of a dynamical system with a higher dimensional phase space.

For concreteness, let us consider a deterministic, continuous dynamical system with a three-dimensional phase space, so that its evolution is given by differential equations. In general, the phase space trajectory of the system is quite complicated. Henry Poincaré, the great French mathematician noticed (towards the end of the nineteenth century) that it is advantageous to focus attention on a two-dimensional *section* (intersection of the three-dimensional phase space trajectory with a two-dimensional plane in the phase space) rather than the entire trajectory (see *Figure 1*). To appreciate the significance of the idea of *Poincaré sections*, let us follow the trajectory and mark the successive intersections of the trajectory with a plane parallel to the XY-plane each time it crosses the plane from below. We thus obtain a list $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n), \dots$ (the Z-phase coordinate is fixed). Recall now that this dynamical system is deterministic and continuous, its evolution given by differential equations of the form (1) of part 1. Therefore, given the nth intersection point (X_n, Y_n), the $(n+1)$th intersection point can always be determined. Now, instead of thinking in terms of the continuous evolution, we can think in terms of a *discrete map*

$$(X_{n+1}, Y_{n+1}) = M(X_n, Y_n),$$

which relates successive points on the Poincaré section *directly*.

We thus achieve (a) dimensional reduction from three to two dimensions, (b) replacement of differential equations by discrete algebraic equations, via the use of Poincaré sections. Needless to say, this concept can be generalised to phase spaces of higher dimensions.

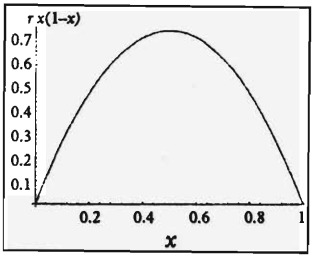


Figure 2. The logistic map.

involves numerical integration. For example, if the Poincaré section of a trajectory consists of a finite number of intersections as $n \to \infty$, the motion must be periodic. If the motion is quasiperiodic or chaotic, then points on the Poincaré section are not a finite set; in the former case they lie on a low-dimensional surface, and in the latter, they fill the $(N-1)$ dimensional space.

The best studied dissipative mapping is undoubtedly the logistic map, shown in *Figure 2*.

$$x_{n+1} = M(x_n) \equiv r\, x_n\,(1 - x_n) \qquad (2)$$

where $0 < x_n < 1$ and $1 < r < 4$. Even though it does not correspond to a Poincaré section of a two-dimensional system (since the mapping is not invertible i.e. if we solve for x_n given x_{n+1}, we get two values, which is not possible in differential equations as they have unique solutions), it does show most of the interesting properties of chaotic systems.

To answer the questions asked above, one would like to know whether for a particular value of r, a periodic orbit of length p, i.e. p points $(x_0, x_1, \ldots, x_{p-1}, x_p \equiv x_0)$ repeating themselves again and again, exists or not. And further, if it exists, whether it is attracting or not. Such a periodic orbit would exist if there are real solutions of the equation

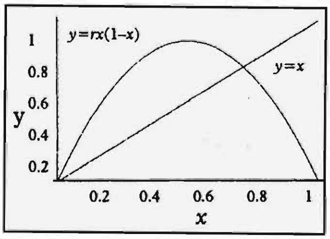
$$M^p(x) \equiv M(M\ldots(\,p \text{ times})\, M(x)\ldots) = x. \qquad (3)$$

If one plots the curve $M^p(x)$, the intersections with the diagonal line $y = x$ correspond to real solutions of (3). It can be shown that these orbits are stable and attracting if

$$|\, M'(x_0)M'(x_1)\ldots M'(x_{n-1})| \leq 1$$

where $x_0, x_1, \ldots, x_{p-1}$ are the solutions of (3). For low order periodic orbits, the analysis is straightforward, and can be carried out easily. The results can be summarised as follows.

Figure 3. Geometric determination of period 1 fixed point.



When $1 < r < 3$, the period one orbit exists ($x_0 = 1 - 1/r$) and is stable, since the slope at $(1 - 1/r)$ is $2 - r$. This can be seen in

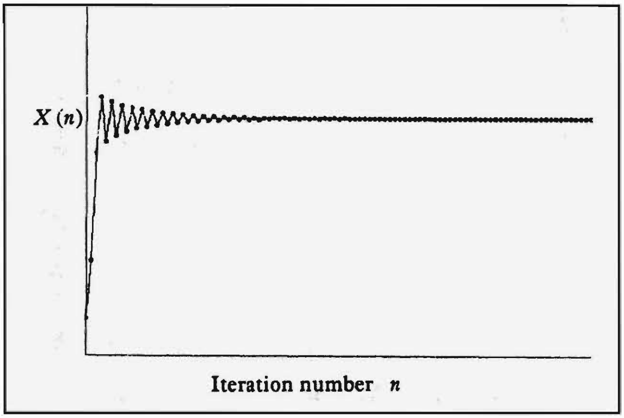


Figure 4. Time series of logistic map with 1 < r < 3.

Figure 3 , where the point of intersection is $(1 - 1/r)$, and the time evolution is shown in *Figure 4*. The period two orbit does not exist, i.e., there is no real solution to (3) for $p = 2$. At $r = 3$ however, the period one orbit becomes unstable as $2 - r = -1$, and a period two orbit is simultaneously created and is stable (*Figures* 5, 6). This is known as a *period doubling bifurcation*. Similarly at $r \approx 3.4$ the period two orbit becomes unstable and a stable period four orbit is created. Let us denote r_n as the value of r when a period 2^n orbit is created (and a period 2^{n-1} is destroyed.) The process of such bifurcations goes on and on for smaller and smaller increments in r and at a limiting value $r \approx 3.57$ the period becomes infinite. This is the *period-doubling cascade*: period 1 gives way to period 2, which gives way to period $4 \equiv 2^2$, then period $8 \equiv 2^3, \ldots$ to period 2^n with $n \to \infty$, and thereby to chaos. This 'scenario' is best illustrated in the bifurcation diagram in which one plots the attracting set the map evolves to

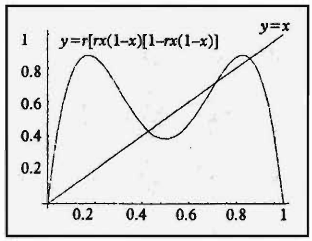


Figure 5. Geometric determination of period 2 fixed points.

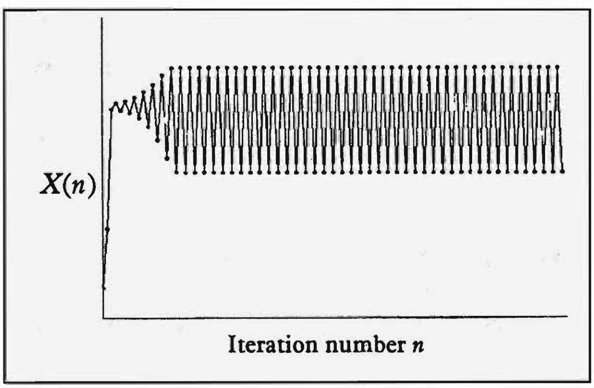


Figure 6. Time series of logistic map for period 2.

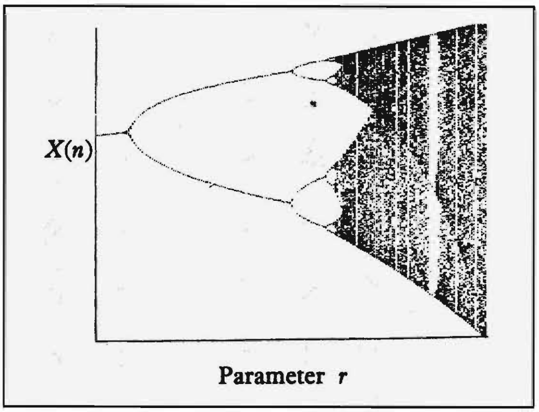


Figure 7. Bifurcation diagram.

with the parameter r, as seen in *Figure* 7. The rate of convergence, namely

$$\delta_n \equiv \frac{r_n - r_{n-1}}{r_{n+1} - r_n}$$

goes, in the limit of $n \to \infty$, to a universal number, the Feigenbaum constant $\delta = 4.6692016\ldots$. This value holds for all systems which exhibit chaos (through period doubling) due to a leading quadratic non-linearity. (For different order non-linearities, the value of δ changes, even though the period doubling *route* to chaos is followed: the fact of a geometric scaling is a universal property.)

For this map, for a particular value of r, there can be at most *one stable periodic orbit* which is attracting. Of course, at a given value of r there need be no stable periodic orbits at all – this is when the motion is truly chaotic. At the same time, periodic orbits of arbitrary periods exist, beyond the point of accumulation of the period 2^n orbits, namely for $r > 3.57$. Each new periodic orbit is created abruptly, either through a period-doubling bifurcation as described above, or through a *tangent* bifurcation to be described in the next para.

If one looks at the plot of $M^p(x)$ and $y = x$, just before the creation of such an orbit, what is observed is that there is no intersection, and then as r is increased, a local minima moves down and becomes tangential to the diagonal, causing an

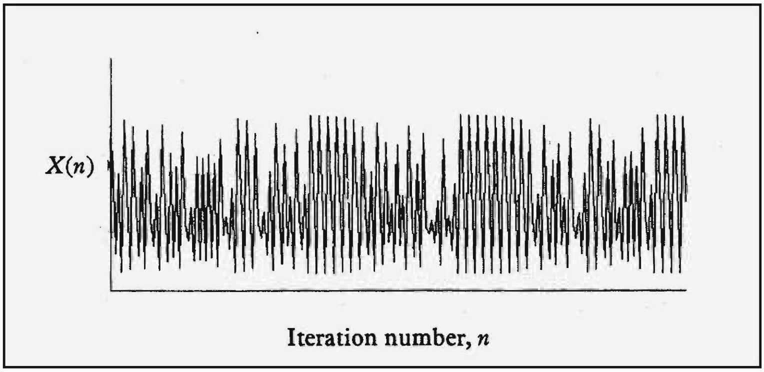


Figure 8. Time series showing intermittent behaviour before the creation of period 3.

intersection at two points. Stability analysis shows that of the two period p orbits so created, one is stable while the other is unstable. This bifurcation is the tangent or *saddle-node* bifurcation. Just prior to the bifurcation, that is just before tangency, there is no stable periodic orbit. However, any point that falls near it will spend very long times near the 'apparent' fixed points, while the motion away from this point will continue to be chaotic. Thus just before a tangent bifurcation one will see bursts of chaotic motion between long periods of apparently periodic motion, seen in *Figure 8*. This is known as *intermittency*, and is commonly seen in a variety of experimental situations.

This occurrence is also reflected in the bifurcation diagram with the sudden appearance of a periodic window. The period 3 window is most prominent as seen in *Figure 9*.

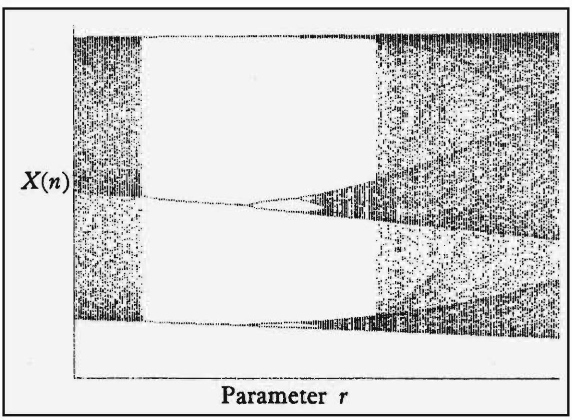


Figure 9. Bifurcation diagram showing the period 3 window for $r > 3.83$.

Box 2. Fourier Transform/Power Spectrum

Any periodic function satisfying certain general conditions may be written as a trigonometric series:

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos \omega_n t + b_n \sin \omega_n t) \qquad (4)$$

where, $\omega_n = 2n\pi/T$ are the frequencies and T is the time period of the function.

Here if most of the a_ns and the b_ns are zero, the set of frequencies needed to describe the function would be finite, but otherwise would be countably infinite (one-to-one correspondence with the set of integers n).

It is also possible to similarly decompose an aperiodic function, where the summation becomes an integral, the set of frequencies is the set of real numbers R (instead of n) and the coefficients a_ns and b_ns become a complex function $g(\omega)$ called the Fourier Transform of $f(t)$.

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(\omega)\, e^{i\omega t} d\omega \qquad (5)$$

If $f(t)$ is known $g(\omega)$ may be calculated as

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t} dt \qquad (6)$$

As $g(\omega)$ is complex, one generally looks at the *power spectrum* $P(\omega) = |g(\omega)|^2$. If the power spectrum of a periodic function is calculated, it will be found to be non-zero only at a finite or countably infinite number of points, *i.e.* it will be discrete. If however it is calculated for an aperiodic function, the number of frequencies will be infinite, *i.e.* the spectrum will be continuous. This serves as a very effective test to determine if some motion is chaotic or not, as the continuous nature of the spectrum is a necessary condition for chaos.

The Ruelle–Takens Scenario

It has been seen in the previous section that one 'route' from periodic to aperiodic (chaotic) motion is that of successive period doublings. Another route is suggested by the nature of the power spectra of periodic and aperiodic motion.

It is well known that a periodic function may be written as a trigonometric series with the frequencies of the sine and cosine

Suggested Reading

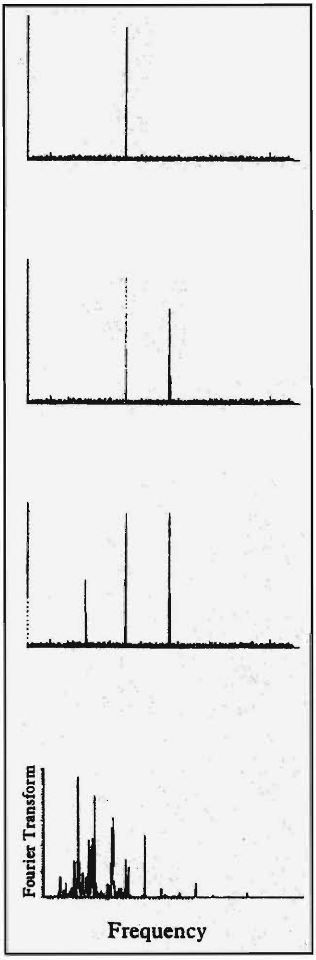
- M Lakshmanan and K Murali. *Chaos in Nonlinear Oscillators: Controlling and Synchronization*. World Scientific. Singapore, 1996.
- T L Carroll. *American Journal of Physics*. 63. 377, 1995.

components either a finite or a countably infinite set (see *Box 2*). However it is found that functions which do not repeat require an uncountably infinite set of frequencies to describe them.

Thus it follows that any transition from periodic to aperiodic motion requires an addition of an infinite number of frequencies to the system (or more accurately, addition of sine or cosine functions, with the new frequencies). The power spectrum thus goes from being discrete to being continuous.

In the period doubling route, at each period doubling bifurcation a subharmonic frequency is added (for each frequency present), which gives in the limit, an infinite number of them, leading to aperiodic motion. Immediately it can be seen that another route might be possible – that of adding *one frequency at a time* instead of doubling the time period. Such bifurcations exist and are known as *Hopf bifurcations*. And an infinite succession of such bifurcations would lead to chaos. Such a scenario had been suggested by Landau for the onset of turbulence in fluids. However, in the early 1970's, Ruelle and Takens showed that after a finite number of Hopf bifurcations (3 or even 2), the dynamics can become intrinsically chaotic and trapped in a so-called 'strange attractor', namely a region of phase space with a complicated topology, characterised by fractal geometry. Thus it is found that in this new route, only a finite number of Hopf bifurcations is required to go into chaos instead of the infinite number we expected earlier.

Figure 10. Power spectrum after the Hopf bifurcations resulting in chaos.



Such a process can be studied by looking at the fourier transform or the power spectrum where distinct peaks corresponding to the new frequencies appear. This phenomenology for a typical situation is shown in *Figure 10* in the power spectra after the first few Hopf bifurcations.

So a given dynamical system can adopt one or the other of the two pathways to chaos. In the next part we describe laboratory experiments on chaos, employing electronic circuitry.

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