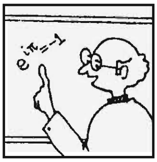

In this section of Resonance, *we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.*

Special Relativity – An Exoteric Narrative

Wherein We Put Formulas in Their Place!

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Some time earlier[1] we treated our readers to an exoteric narrative of special relativity's kinematics in which we shunned the use of mathematical formulas. However, the simple and concise account presented then has left quite a few questions open, and we now present the promised sequel to supply the requisite answers.

[1] See *Resonance*, Vol.3. No.1. 61–62, 1998.

Making the Train and the Tunnel Infinitely Long

Let us note at the very outset that it makes things simpler to extend both the tunnel and the train to an indefinite length rightwards as well as leftwards. Further, we can place a whole continuum of observers in the tunnel as well as aboard the train to ward off incommoding gaps along their majestic stretch.

Consider now a fixed watchman M in the tunnel. At each point of time, some passenger M′ necessarily appears against M. At this precise instant, M can take a look at an arbitrary fellow-watchman L with his left eye, and another arbitrary fellow-watchman N with his right eye. When he does so, he has perforce to sight one or other passenger L′ against L, and one or other passenger N′ against N:

We set length [LN] = a, length [LM] = ap, length [L′N′] = b, and length [L′M′] = bq. If we write

$$P = 1 - p \quad \text{and} \quad Q = 1 - q, \qquad (1)$$

it readily follows that length [MN] = aP and length [M′N′] = bQ.

We assume that all these lengths have been measured by tunnel-dwellers. Moving lengths are measured by noting the coincidences of their extremities with points fixed in the tunnel, and subsequently applying appropriate corrections for any possible non-simultaneities of such coincidences. No inconsistencies can arise in the process. For if they did, these would be indicative of different parts of the train moving with different velocities at different times – contrary to the assumed uniformity of the entire train's motion on an everlasting basis.

Some Poetic Visions

It is easy to be misled by *Figure 1* into thinking – for instance – that a=[LN]=[L′N′]=b. However, as judged by M, the coincidences (L′:L), (M′:M) and (N′:N) have actually occurred at *different* times. Conclusions such as a = b and ap = bq are

Figure 1.

```
|.................................................. b ..................................................|
>............................bq................................................|..........bQ............>
  L'                                                                             M'                 N'
        ---*----------------------------------------------------------------*--------------------*---
        ---*----------------------------------------------------------------*--------------------*---
  L                                                                              M                  N
  :                                                                              :                  :
|...............................ap.............................................|..........aP.............|
|.................................................. a ..................................................|
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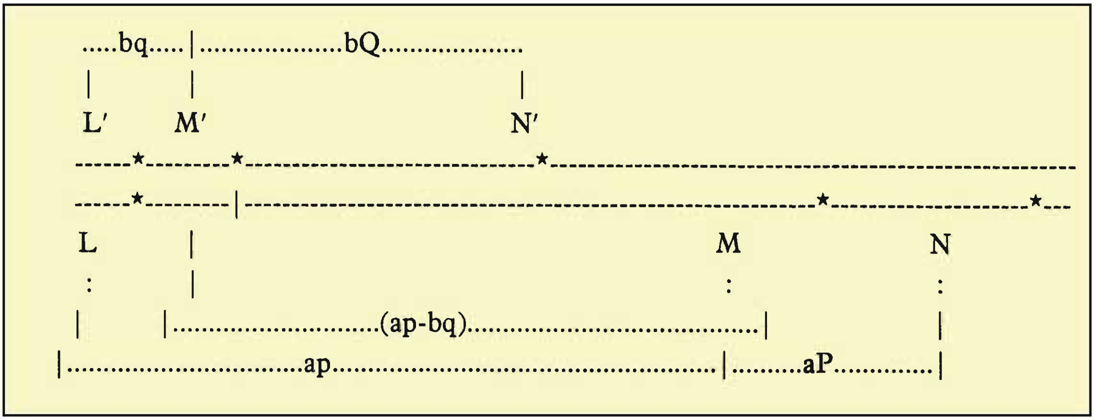

Figure 2.

therefore quite erroneous. As a matter of fact, M considers it a safe bet that at the exact instant of time when the coincidence event (L':L) happened, the watchmen and the passengers under surveillance were configured as shown in *Figure 2*.

Unlike in connection with *Figure 1*, the depiction of *Figure 2* cannot be seen by any watchman with his actual eyes (or photographed). Poets, though, declare with gusto that watchman M does see the tidings of *Figure 2*, but soon let him down badly by adding that the 'seeing' is done with his 'mind's eye'!

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Suppose that, as against light rays darting across with a speed c, the train coasts ahead with a smaller velocity u. We assume of course that it is the tunnel-dwellers who have measured c and u. Now watchman M referring to *Figure 2* tells us that during the time passenger M' with his speed u traversed the segment shown with the length-label (ap bq), light signals from (L':L) leaped across the longer distance ap=[LM] to close in on him. Accordingly,

$$(ap \quad bq)/(ap) = u/c \qquad (2)$$

Next, there is also the other coincidence event (N' :N) to contend with. Again, the mind's eye of watchman M conjures up a 'view' of how the sixsome of observers must have configured themselves at the exact instant of time when this event took place (*Figure 3*):

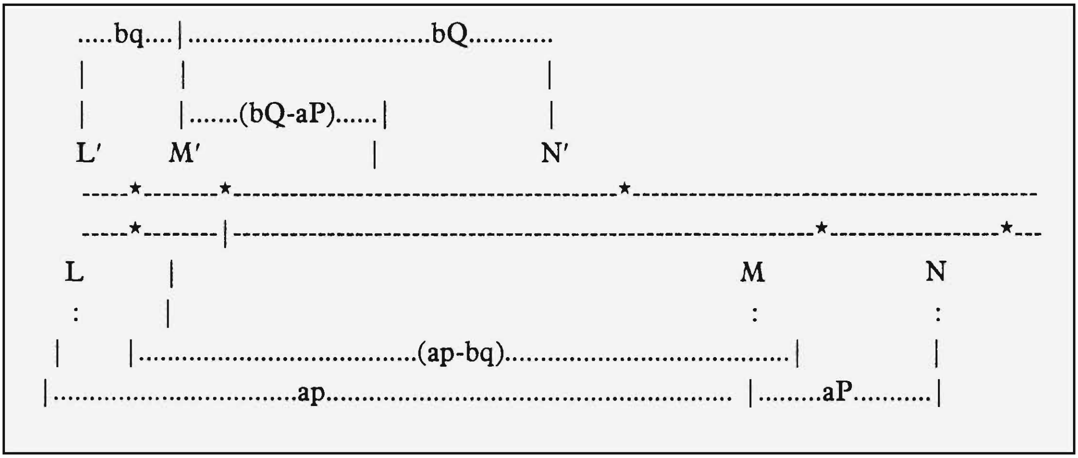

Figure 3.

This in a like manner leads to

$$(bQ - aP)/(aP) = u/c. \qquad (3)$$

It will be felicitous to write

$$\tilde{u} = u/c \qquad (4)$$

for the ratio of the train's speed to the speed of light. Readers can now easily verify the following consequences of (1) through (4):

$$\tilde{u} = (p - q)/(p + q - 2pq) \qquad (5)$$

$$b/a = 2p\,(1 - p)/(p + q - 2pq) \qquad (6)$$

$$1 + \tilde{u} = 2p\,(1 - q)/(p + q - 2pq) \qquad (7)$$

and

$$1 - \tilde{u} = 2q\,(1 - p)/(p + q - 2pq). \qquad (8)$$

The Train's Precise Speed in our Narrative

In our simplified narrative recounted earlier, we had [W0], [W4], [W5], [P0], [P1] and [P5] in place of L, M, N, L′, M′ and N′ respectively. The correspondence suggests the values $p = LM/N = 4/5$ and $q = L'M'/L'N' = 1/5$. Inserting these values in (5) and (6) we find $\tilde{u} = u/c = 15/17$ and $b/a = 8/17$. The railroad train in the narrative was therefore racing ahead with 15/17ths of the speed of light, and, as judged by watchman [W4], it was only 8/17ths as long as the tunnel.

What happens if the narrative is modified so that [P3] appears against [W1] at the crucial moment (other things being the same)? We should then use $p=1/5$ and $q=3/5$. The train now coasts leftwards (why?) with 5/7ths the speed of light, and [W1] finds the lengths of the train and the tunnel to be standing in the ratio 4:7.

Reappearance of Relativity

In our transactions so far, relativistic considerations have been conspicuous by their total absence. Relativity does come in, however, when the complete equivalence of the train and the tunnel is asserted They are equally valid frameworks within which natural phenomena can be described. In particular, relativity claims that it is perfectly legitimate to *interchange* the roles of the tunnel and the train in our equations. Thus, referring back to *Figure 1*, passenger M′ can independently make his own measurements of the various lengths and velocities involved, and possibly come up with quite different values. We designate these different values by tagging the corresponding erstwhile variables with primes, a', b', p', $u' \sim$ etc. (u' being the velocity of the tunnel relative to the train). Special relativity now consists in the affirmation that (5) through (8) must stay valid when they are appropriately recast with primed variables to reflect the standpoint of passenger M′, too. In this connection, we recall our description of *Figures 2 and 3* as 'poetic'. The idea is to drive home a little plain prose – that the corresponding constructions of the 'mind's eye' of passenger M′ would considerably differ from the depictions of *Figures 2* and *3* in many details.

At this point, a rather pleasing windfall comes our way. Suppose that somewhere on the train the passengers have marked off their chosen unit of length, and that when the tunnel-dwellers measure the marked-off length they obtain a certain value G.

Now it is reasonable to assume that the latter would obtain the same value G irrespective of when they measure the length in question, and also irrespective of in which part of the train the passengers have marked off their unit length. We shall call this assumption the 'G-hypothesis' of space-and-time homogeneity. It can easily be inferred from this hypothesis that both tunnel-dwellers and train-denizens must necessarily obtain identical values for the ratios in which any particular segments of the train – and, by the same token, any particular segments of the tunnel as well – are internally subdivided. As such, $p'=p$, $P'=P$, $q'=q$ and so forth, and there is no need to use primes against p, q, etc. (In fact, we should admit that our numerical computation of the train's speed in the preceding section anticipated this result somewhat on the sly: what passed off as $q=1/5$ there was actually $q'=1/5$.)

The Genesis of Length-Contraction

Windfall tucked in, some examples of how equations transform when we switch from the reasonings of watchman M to those of passenger M' may now be considered. The simplest to explore is (5). When recast to express the standpoint of M', this leads to the not very surprising (?) result $\tilde{u}' = -\tilde{u}$ Can you see how?

Our main interest, however, is in (6):

$$b/a = 2p(1-p)/(p+q-2pq) \qquad (9)$$

When we pass over to the standpoint of M', this formula takes on the form

$$a'/b' = 2q(1-q)/(p+q-2pq). \qquad (10)$$

Multiplying Eqs. (9) and (10) and using Eqs. (7) and (8),

$$(a'/a)(b/b') = 1-\tilde{u}^2 = g^2, \text{ say.} \qquad (11)$$

There is quite a world of difference of nature between a'/a and b/b' on one hand and b/a and a'/b' on the other. In the latter instance, only a single observer each time has the charge of both the numerator and the denominator, and therefore whichever units of length he employs simply cancel out. Do look next at a'/a for the contrast. Here it is watchman M who measures the denominator, and passenger M′ who takes on the numerator. Each of these observers has all along been entirely free to employ any arbitrary unit he likes to measure lengths. But where physicists have to be constantly dealing with diverse moving frames, it makes life simpler to get the observers in these frames to use (in some reasonable sense) the 'same' measuring units. Let us persuade M and M′ to do so, then. If now either of a'/a and b/b' were to exceed the other, our kinematics would be in a position to distinguish between the tunnel and the train, and the relativistic symmetry of the frames would be lost. This forces the inference

$$a'/a = b/b' = g, \quad (g^2 = 1 - \widetilde{u}^2). \qquad (12)$$

From here we may eliminate a' and b by using (9) and (10). It then follows that (12) not only implies but is implied by

$$p(1 \quad p)a^2 = q(1 - q)b'^2. \qquad (13)$$

This relation points to a practical recipe for realising the 'same' unit of length in the tunnel and aboard the train. Just pick out some arbitrary constant $H > 0$, then get watchman M to measure his a, and passenger M′ his b', in such respective units as make the left and the right members of (13) both equal to H – and you're done. (Alternative prescriptions are possible, too.)

Equation(12) retrieves the famous length-contraction relation of Lorentz and Fitzgerald. Our treatment has concerned itself with just a particular set of lengths linked in a highly specialised fashion to the (M′:M) coincidence event. Nevertheless, the 'G-hypothesis' of homogeneity we alluded to a little earlier does guarantee the validity of (12) for arbitrary sets of lengths not necessarily connected in any special way with the (M′:M) event.

Having chosen their respective units of length in the manner just detailed, watchman M and passenger M′ can obviously go on to choose their units of time, too, so that the omnidirectional

The scifi heroes make it to the remote galaxies and return home well within their lifetimes only because, with ultra-high speeds totally at their disposal, the distances of their destinations can be as drastically length-contracted as convenient.

speed of light in vacuo assumes a common numerical value c as measured by both of them. Explicit prescriptions are thus available for realising what are called the 'same' units of length and time in different frames. The naive idea of actually transporting measuring rods and time-pieces from the tunnel into the train, however, is hardly acceptable since the accelerations inflicted on the standards in the process could well wreak havoc on them.

Accosting us next, of course, are the time-dilations. These, however, are reincarnations of length-contractions in a modified context. The scifi heroes make it to the remote galaxies and return home well within their lifetimes only because, with ultra-high speeds totally at their disposal, the distances of their destinations can be as drastically length-contracted as convenient. There is thus no need to dwell separately on time-dilations.

A Spacelike Interval and its Invariance

It would be an easy exercise to go over from (12) to the topic of one-dimensional Lorentz transformations, but we do not propose to go into any details here. The connection (13) with the relativistic (spacelike) interval between the events (L′:L) and (N′:N), however, is too striking to be skipped over at this point. Referring back to *Figure 1* (and remembering that $P=1-p$), we readily see that the left member of (13) (a p a P) is just the product of the lengths of the two segments [LM] and [MN] as measured by watchman M. Furthermore,

$$\begin{aligned} 4\,[LM]\,[MN] &= [LM + MN]^2 - [LM - MN]^2 \\ &= [LN]^2 - [LM - MN]^2 \end{aligned}$$

[LN] expresses the separation-in-space of the events (L′:L) and (N′:N), and [LM–MN] is c times the separation-in-time of the same two events, both separations being as measured by M. Accordingly, the left member of (13) expresses the quantity

$$\{(\text{spatial separation})^2 - c^2\,(\text{temporal separation})^2\}/4 = c^2\,T^2/4, \qquad (14)$$

say, for the pair of events in question as measured by M. The right member of (13) likewise expresses the same quantity for the same pair of events, but as measured by M′. The quantity T defined by (14) is called the spacelike interval between (L′:L) and (N′:N). What (13) tells us, therefore, is that this interval is invariant.

Addition of Velocities

Lastly, we present an interesting derivation of the law of addition of velocities using our approach. To this end, we bring in a second idealised train which races through the same tunnel of ours on a parallel track, thus:

The second train carries the passengers L″, M″ and N″, and each of these is involved in a triple coincidence event as indicated in *Figure 4*. The view here is strictly local to the site (M″:M′:M) — except for the length-labels which are supposed to reflect watchman M's measurements. Thus *Figure 4* is no more than a simple augmentation of *Figure 1*.

Let us now go back to (7) and (8). Dividing the first of these by the second,

$$(1+\tilde{u})/(1-\tilde{u}) = p(1-q)/\{q(1-p)\}.$$

By the same token, if the second train's speed is $v = c\tilde{v}$ relative

Figure 4.

```
>......................................dr..........................................|..........dR.............>
L”                                                                                 M″                   N″
  -----*------------------------------------------------------------------*------------------------*----
>......................................bq.........................................|...........bQ............>
L’                                                                                 M′                   N′
 -----*---------------------------------------------------------------------------*------------------*
 -----*---------------------------------------------------------------------------*------------------*
L                                                                                  M                    N
:                                                                                  :                    :
|......................................ap.........................................|............aP............|
```

to the first train, and $w=c\tilde{w}$ relative to the tunnel,

$$(1+\tilde{v})/(1-\tilde{v}) = q(1-r)/\{r(1-q)\}$$

and

$$(1+\tilde{w})/(1-\tilde{w}) = p(1-r)/\{r(1-p)\}.$$

These imply

$$(1+\tilde{w})/(1-\tilde{w}) = \{(1+\tilde{u})/(1-\tilde{u})\}\{(1+\tilde{v})/(1-\tilde{v})\}.$$

From this it follows

$$\tilde{w} = (\tilde{u}+\tilde{v})/(1+\tilde{u}\,\tilde{v}),$$

which is the law of addition of velocities in the one-dimensional space of tunnel's interior.

Errata: In Part 1 of this article which appeared in Vol.3, No.1 on page 62 under section 'The Penultimate Watchman's Findings', the figure should appear as given:

P				P	P
0				(?)	5
*				*	*
				...>=<...	
*	*	*	*	*	*
W	W	W	W	W	W
0	1	2	3	4	5

"Science is above all a world of ideas in motion. To write an account of research is to immobilize those ideas; to freeze them, like describing a horse race with a snap-shot. It is also to transform the very nature of the research; to formalise it. ... In short, writing a paper is to substitute order for the disorder and agitation that animate life in the laboratory."

Francois Jacob