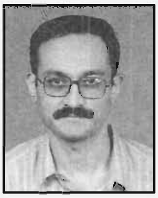
On Kepler's First Law

The Law of Ellipses

Shailesh Shirali



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Kepler analysed the observations of the planets to discover the elliptical shape of planetary orbits. The mathematical connection between the shape of the orbit and the law of force came with Newton. This article explores both Newton's general approach and later simplifications.

Introduction

This article is about Kepler's first law or the law of ellipses, named after its discoverer and shown by Newton to follow from an inverse square law of force: *A planet revolving about the sun moves in an elliptical orbit with the sun at one focus of the ellipse.* We present two proofs of this classical result: Newton's proof, from the *Principia Mathematica*, and Feynman's proof, from *Feynman's Lost Lecture* (see D Goodstein and J Goodstein in Suggested Reading). (The latter proof has been discovered independently and at different times by others; e.g., by Hamilton and Maxwell. See A Lenard in Suggested Reading for an unusual treatment.)

Preliminaries

Given two points S and H and a constant a such that $2a > SH$, the ellipse with foci at S and H and semi-major axis equal to a is defined to be the locus

$$\{P : PS + PH = 2a\}.$$

The mid-point C of SH is the *centre* of the ellipse. A chord PG of the ellipse which passes through C is a *diameter* of the ellipse; and if DK is the diameter parallel to the tangent to the ellipse at P, then DK and PG are *conjugate diameters*.

The lemmas listed below refer to a fixed ellipse $\mathcal{E}$ with centre

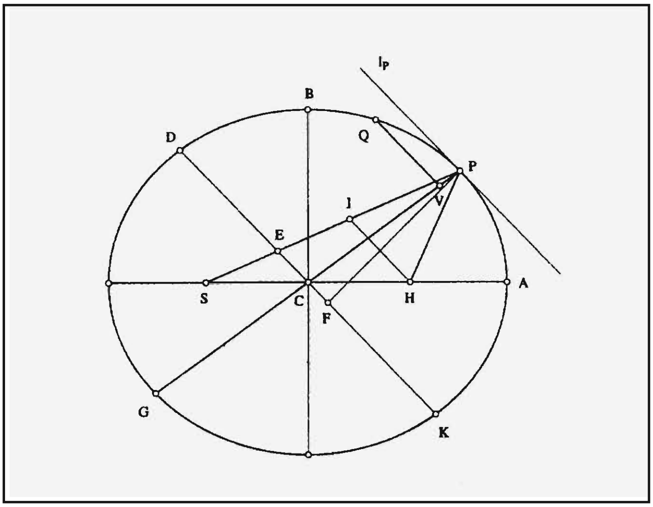


Figure 1. Standard results concerning an ellipse.

C, foci S and H, semi-major axis $CA = a$ and semi-minor axis CB; see *Figure 1*. For an arbitrary point P on $\mathcal{E}$, the tangent to the ellipse at P is ℓ_P and the diameter of the ellipse through P is PG.

Following standard convention, we denote vectors with an arrow above the letters.

Lemma 1. *The area $|\vec{C}\vec{P} \times \vec{C}\vec{D}|$ of the parallelogram spanned by $\vec{C}\vec{P}$ and $\vec{C}\vec{D}$ is constant for all pairs of conjugate diameters PG and DK*

Proof. The simplest proof is via affine projection of the ellipse to a circle. The diameters PG and DK map to a pair of perpendicular diameters of the circle, and the area of the parallelogram spanned by $\vec{C}\vec{P}$ and $\vec{C}\vec{D}$ is now the same for all pairs of diameters. Since all areas change by the same scale factor under an affine map, the result follows.

Lemma 2. *Let PG and DK be a pair of conjugate diameters of the ellipse, and let PS meet DK at E. Then $PE = a$.*

Proof. Referring to *Figure 1*, where HI is parallel to KD, we have: $PI = PH$ (by Lemma 4, a standard result); $SE = EI$, because $SC = CH$; $PS + PH = 2a$ (by definition). These relations yield $PE = a$. (Note: The lemma is Newton's own.)

Note that all that the proof requires is that the force be a central one; the inverse square nature of the force plays no role whatever.

Lemma 3. *Let PG be any fixed diameter of $\mathcal{E}$, and let Q be a variable point on $\mathcal{E}$. Let the line through Q parallel to ℓ_P meet PG at V Then $QV^2/(PV\ \ VG) = DC^2/PC^2$.*

Proof. Suppose that DK is perpendicular to PG. Let the equation of $\mathcal{E}$ with respect to CP and CD as the x- and y-axes be $x^2/p^2 + y^2/d^2 = 1$; let $V = (v, 0)$. Then $PV\ \ VG = p^2 - v^2$, and

$$\frac{QV^2}{PV \cdot VG} = \frac{d^2\left(1 - v^2/p^2\right)}{p^2 - v^2} = \frac{d^2}{p^2} = \frac{DC^2}{PC^2}.$$

If DK is not perpendicular to PG, we use a shear with axis GP to transform the figure to one where the image of DK is perpendicular to GP (which by definition remains invariant under the shear). Let the image of an arbitrary point X be denoted by X'; then $D'C'$ and $Q'V'$ are both perpendicular to GP. Writing θ for the angle between CD' and CD, the following equality holds:

$$\frac{Q'V'}{QV} = \cos\theta = \frac{D'C'}{DC};$$

so QV/DC remains invariant through the shear. It follows that

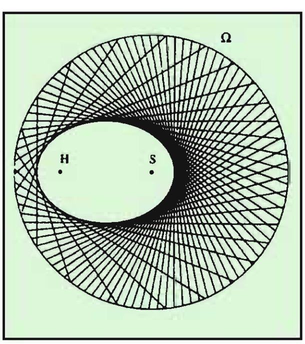
$$\frac{QV^2}{DC^2} = \frac{Q'V'^2}{D'C^2} = \frac{PV \cdot VG}{CP^2},$$

as required.

Lemma 4. (a) *For any arbitrary point P on $\mathcal{E}$, the focal radii SP and HP make equal angles with ℓ_P.* (b) *The locus of images of H under reflection in a variable tangent ℓ_P to the ellipse is a circle with radius $2a$, centred at S.*

(Result (a) is well-known; for a nice 'variational' proof by Feynman. See D Goodstein and J Goodstein in Suggested Reading. Result (b) follows from (a) and the relation $PS + PH = 2a$. The latter result allows for a pretty paper folding exercise. From a sheet of wax paper, cut out a circular disk Ω with centre S, and mark a point H close to its periphery. Repeatedly fold points on the periphery of Ω upon H, making a neat crease each time. After working one's way around the periphery, the creases are seen to envelope an ellipse with foci at S and H; see *Figure 2*.)

Figure 2. Paper folding construction of an ellipse.



Remark. Characteristically, Newton waves away these lemmas with the comment, "This is demonstrated by the writers on the conic sections."

Newton's Proof of the Area Theorem

Before presenting Newton's proof of the ellipse theorem, we briefly present his proof of the area theorem: *The areas which revolving bodies describe by radii drawn to an immovable centre of force ... are proportional to the times in which they are described.* For the proof we discretise the situation. In *Figure 3*, S is the immovable centre of force, and A, B, C, D ... are the positions of the revolving body after equal intervals of time Δt. Imagine that the force acts in 'bursts' (at the points $A, B, C, D, \ldots$). At B the force induces a fall $\vec{B}\vec{V}$, as a result of which the body moves to C; in the absence of the force its inertia would have carried it to c, say. Observe that $\vec{A}\vec{B} = \vec{B}\vec{c}$ and that $CcBV$ is a parallelogram. It is immediate that the area of triangle CSB is equal to that of triangle BSA, for both are equal in area to triangle cSB. In the same way we see that triangles DSC and CSB have equal area, and so on. Letting $\Delta t \to 0$, the law of areas follows and Kepler's second law is explained. Note that all that the proof requires is that the force be a central one; the inverse square nature of the force plays no role whatever.

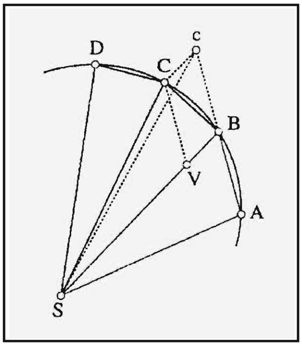


Figure 3. Proof of the area theorem.

Newton's Proof of the Law of Ellipses

One notes with surprise that questions have been raised of late concerning Newton's treatment of the proposition that an inverse square law of force implies a conic section orbit. (See R Weinstock; B Pourciau in Suggested Reading for details.) Indeed one finds in the *Principia* that Newton actually addresses the converse proposition. The famous proposition XI poses the problem thus: *A body revolves in an ellipse; it is required to find the law of centripetal force tending to the focus of the ellipse.* (Note the precise and packed nature of the sentence, a style of writing at once characteristic of Newton.) This is followed by an elegant proof showing that the law of force must be an inverse square one. In the first edition of the *Principia*, Newton left it at this, but inevitably there were protests (one came from Johann Bernoulli); and a later edition carried a brief addendum which in essence is a uniqueness argument; it shows

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how, arguing from curvature considerations (note that curvature is a locally defined quantity), one can indeed conclude that orbits under an inverse square law of force are conic sections. With the addendum in place there can be no further controversy about the proof, and we shall not dwell upon the matter.

A word needs to be added regarding Newton's style of writing. Several writers have remarked on the 'icy grandeur' of the *Principia*, and dense and packed the text certainly is. Chandrasekar remarks (See S Chandrasekhar in Suggested Reading) on the 'directness, absence of superfluity, and the entirely elementary character' of Newton's proofs, and on how nevertheless these tend to be obscured by his style of writing mathematical derivations in continuous prose. Newton may have deliberately chosen this style. Indeed he is reported to have told a friend that

> "... to avoid being bated by little smatterers in mathematics [he] designedly made [his] principle abstruse; but yet so as to be able to be understood by able mathematicians."

(The reasons for this are complex and have much to do with the unpleasant priority battles Newton had with Hooke and later with Leibnitz; but this need not concern us here.) Present-day readers certainly do find the *Principia* heavy going, but this owes in part to changing fashions in mathematics education. Students nowadays are exposed very little to 'old fashioned geometry', for instance the geometry of the conic sections, and Newton's arguments tend to rely heavily on elementary geometry.

Two elementary principles used repeatedly and very effectively by Newton are [P1] (he refers to this as 'Galileo's principle') and [P2] as given below.

[P1] Under an acceleration of magnitude a directed towards a point S, the distance s which a particle falls towards S in a small time interval t is given by $s = \frac{1}{2}at^2$.

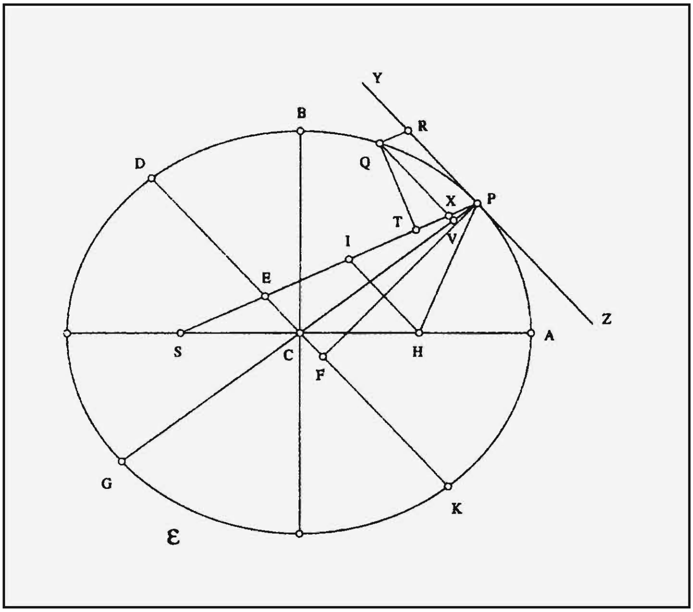


Figure 4. Planet in an elliptical orbit.

[P2] For a particle in a curved orbit, at any instant its normal acceleration (directed towards the instantaneous centre of curvature) is given by v^2/r, where r is the instantaneous radius of curvature and v the tangential velocity at that instant.

We proceed to give Newton's proof. The diagram employed by him is reproduced in *Figure 4*, except that (with due apologies) we have replaced the labels x and v by X and V respectively. The figure shows an elliptical orbit $\mathcal{E}$, with centre C and foci S and H; it is supposed that forces are directed towards S (the 'sun'). For an arbitrary point P on $\mathcal{E}$, let PG and DK be the associated conjugate diameters (DK is parallel to the tangent YZ to $\mathcal{E}$ at P). Let the planet move from P to Q in a small interval of time t. Completing the parallelogram $PRQX$, with R on PY and X on PS, we view the movement $\vec{PQ}$ as composed of a movement $\vec{PR}$ due to inertia, which would have taken place in the absence of attraction towards S, and a 'fall' $\vec{PX}$ resulting from the central force. Invoking [P1], we take the acceleration at P towards S to be measured by the quantity $2PX/t^2$ or $2QR/t^2$. From the law of areas we know that t is proportional to the area of sector QSP. This area is approximately equal to the area of triangle QSP, which equals $SP\ \ QT/2$, where T is the foot of the perpendicular from Q to PS. It follows that the acceleration at P towards S is proportional to

$$\frac{QR}{(SP\ \ QT)^2}. \tag{1}$$

(The constant of proportionality is of no consequence for the moment.)

So everything hinges on the behaviour of QR/QT^2 as Q tends to P along the curve. We shall show that in our situation, the limit of QR/QT^2 as $Q \to P$ does not depend on P; indeed, that the limit equals $1/L$ where $L = 2BC^2/AC$ is the latus rectum of the ellipse. This will imply that the acceleration is inversely proportional to SP^2, which is what was to be proved.

We have, using similarity of triangles and Lemmas 2 and 3,

$$\frac{QR}{PV} = \frac{PE}{PC} = \frac{AC}{PC}, \qquad \frac{PV\ \ VG}{QV^2} = \frac{PC^2}{CD^2}.$$

Again, using similarity of triangles and Lemma 1,

$$\frac{QT}{QX} = \frac{PF}{PE} = \frac{PF}{AC} = \frac{BC}{CD},$$

Next, $QV \approx QX$ since Q is close to P. It follows that

$$\begin{aligned}\frac{QT^2}{QR} &= \frac{QX^2 \cdot BC^2}{CD^2}\ \frac{PC}{PV \cdot AC} = \frac{L}{2}\ \frac{QX^2\ \ PC}{PV\ \ CD^2} \\ &\approx \frac{L}{2}\ \frac{QV^2}{PV\ \ VG} \cdot \frac{VG \cdot PC}{CD^2} \\ &= \frac{L}{2}\ \frac{CD^2}{PC^2}\ \frac{VG \cdot PC}{CD^2} = \frac{L}{2}\ \frac{VG}{PC} \approx L,\end{aligned}$$

since $VG \approx 2PC$. It follows that $QR/QT^2 \to 1/L$ as $Q \to P$, and so the inverse square law of force holds. ∎

Newton now adds the following comments.

> For the focus, the point of contact, and the position of the tangent, being given, a conic section may be described, which at that point shall have a given curvature. But the curvature is given from the centripetal force and the velocity of the body being given; and two orbits, touching one the other, cannot be described by the same centripetal force and the same velocity.

As stated earlier, this is essentially a uniqueness argument. It states that for a particle in an orbit $\mathcal{O}$ under a known central force, if the position and tangential velocity of a particle at a given point P are known, the curvature of $\mathcal{O}$ at P is known (via [P1] and [P2]); and with the focus being given, together with a point P on the curve, the tangent to the curve at P *and* the curvature of the curve at P, a unique conic section $\mathcal{C}$ may be described. The two curves $\mathcal{O}$ and $\mathcal{C}$ now coincide at P upto curvature, a second degree quantity. The fact that the governing differential equation is of the second degree now forces $\mathcal{O}$ and $\mathcal{C}$ to agree everywhere.

We remark here that Newton's analysis yields a simple explanation of Kepler's third law: if $2a$ is the major axis of the orbit (an ellipse, by the second law) and T the orbital time period, then T^2/a^3 is the same for all orbits. Indeed it yields a version of the law which holds for open orbits too (the original version becomes meaningless in this case). Consider again the analysis leading to equation (1). Writing r for the radial distance SP and h for the constant rate at which area is swept out by the radius vector, we see that the acceleration at P towards S is given by $2PV/t^2$ as well as by k/r^2, where k depends only on the mass of the sun and the gravitational constant; and that $h \cdot t \approx \frac{1}{2} r \cdot QT$. Therefore,

$$\frac{k}{r^2} \approx \frac{8 \cdot PV \;\; h^2}{r^2 \cdot QT^2}.$$

Since $PV/QT^2 \to 1/L$ as $P \to Q$, it follows that

$$\frac{L}{h^2} = \frac{8}{k} = \text{constant}; \qquad (2)$$

Newton is clearly in an expansive mood when writing about orbits: we see the mathematician in Newton rather than the scientist!

that is, L/h^2 has the same value for all orbits. *This is Newton's generalisation of Kepler's third law.* Observe that it is meaningful for open orbits. In the particular case when the orbit is an ellipse with a and b as the major and minor axes, we have $L = 2b^2/a$ and $h = \pi ab/T$, where T is the orbital period; so we obtain,

$$\frac{T^2}{a^3} = \frac{4\pi^2}{k} = \text{constant}, \tag{3}$$

and we have recovered the original form of Kepler's third law.

Newton is clearly in an expansive mood when writing about orbits: he examines not only the conditions that imply an inverse square law of force, but also those which would give rise to:

(1) an elliptical orbit, with the force tending to the centre of the ellipse;

(2) an equiangular spiral orbit, with the force tending to the pole of the spiral.

(We see here the mathematician in Newton rather than the scientist!) Here are his findings: in (1) the force varies directly as the distance (acceleration $\propto r$), and in (2) the force varies inversely as the cube of the distance (acceleration $\propto 1/r^3$).

Rather than present Newton's geometrical proofs, we tackle the problems using coordinates. Suppose that the revolving body moves in the xy-plane along a curve $y = f(x)$ under the influence of a force directed towards the origin $S = (0, 0)$. Let $P = (a, f(a))$ and $Q = (b, f(b))$ be neighbouring points on the curve, and let X, T be points on SP such that QX is parallel to the tangent to the curve at P, while QT is perpendicular to SP. Then, from (1), the acceleration at P towards S is given by the expression

$$\frac{1}{SP^2} \lim_{Q \to P} \frac{PX}{QT^2}. \tag{4}$$

Crunching through a few computations, we find that

$$\frac{PX}{QT^2} = \frac{f(a) - f(b) - f'(a)(a-b)}{(bf(a) - af(b))^2} \; \frac{r^3}{f(a) - af'(a)}. \tag{5}$$

(Here $r = SP$.) We need the limit of (5) as $b \to a$; l'Hospital's rule yields:

$$\lim_{Q \to P} \frac{PX}{QT^2} = \frac{r^3 \cdot f''(a)}{2\left(af'(a) - f(a)\right)^3}.$$

Changing notation and writing (x, y) for $(a, f(a))$, y' for $f'(a)$ and y'' for $f''(a)$, we see that the acceleration at P towards S is proportional to the following quantity:

$$\frac{r \cdot y''}{(xy' - y)^3}. \tag{6}$$

Following Newton we present three applications of the basic formula (6).

(i) *Parabolic orbit, with the force directed towards the focus.*

We take the equation of the parabola as $y = x^2 - 1/4$ (observe that its focus is at the origin). Substituting $y' = 2x$ and $y'' = 2$, we find that

$$xy' - y = x^2 + \frac{1}{4}, \qquad r^2 = x^2 + y^2 = \left(x^2 + \frac{1}{4}\right)^2$$

so the acceleration varies as $1/r^2$, implying an inverse square law of force.

(ii) *Elliptical orbit, with the force directed towards the centre of the ellipse.*

Taking the equation of the ellipse as $b^2x^2 + a^2y^2 = a^2b^2$ (the axes being $2a$ and $2b$ respectively), we find that

$$y' = -\frac{b^2 x}{a^2 y}, \qquad xy' - y = -\frac{b^2}{y}, \qquad y'' = -\frac{b^4}{y^3}.$$

This leads to

$$\frac{r \cdot y''}{(xy' - y)^3} = \frac{r}{b^2},$$

so the acceleration varies as r. This agrees with Newton's result.

(iii) *Equiangular spiral orbit, with the force directed towards the pole S of the spiral.*

Feynman chooses to view the system at instants of time when the radius vector has advanced around the orbit through equal angles. In so doing he arrives at the following insight: *As the particle orbits around S, the velocity vector moves in a circle.*

Suppose that the spiral is such that for any point P on the curve the angle between SP and the tangent to the curve at P is 45°. With S as the origin, the differential equation of the spiral takes the form

$$\frac{y' - y/x}{1 + yy'/x} = 1.$$

This leads to the following relations:

$$y' = \frac{x+y}{x-y}, \qquad xy' - y = \frac{x^2}{x-y}, \qquad y'' = \frac{2x^2}{(x-y)^3}.$$

Substituting these into (6), we obtain

$$\frac{r \cdot y''}{(xy' - y)^3} = \frac{2}{r^3},$$

so the acceleration varies as $1/r^3$. Once again, this agrees with Newton's result.

The Maxwell-Feynman Proof of the Law of Ellipses

In his proof of the law of areas, Newton views the system at equally spaced instants of time. Feynman chooses to view the system at instants of time when the radius vector has advanced around the orbit through equal angles. In so doing he arrives at the following insight: *As the particle orbits around S, the velocity vector moves in a circle.*

Feynman's argument has most likely been done using vectors, but he describes it in geometrical language. Let position vectors be measured with reference to the sun S as origin. In a small interval of time Δt, let the position $\vec{r}$, velocity $\vec{v}$ and radial angle θ change by $\Delta\vec{r}$, $\Delta\vec{v}$ and $\Delta\theta$ respectively. Writing $\hat{\vec{r}} = \vec{r}/r$ for the unit vector in the radial direction, we have,

$$\frac{\Delta \mathbf{v}}{\Delta t} = -\frac{k}{r^2}\hat{\mathbf{r}}, \qquad \frac{1}{2}r^2\frac{\Delta\theta}{\Delta t} = h, \tag{7}$$

where k is a constant and h is the rate at which area is swept out by the radius vector. By division we obtain

$$\frac{\Delta\vec{v}}{\Delta\theta} = -\frac{k}{2h}\hat{\vec{r}}. \tag{8}$$

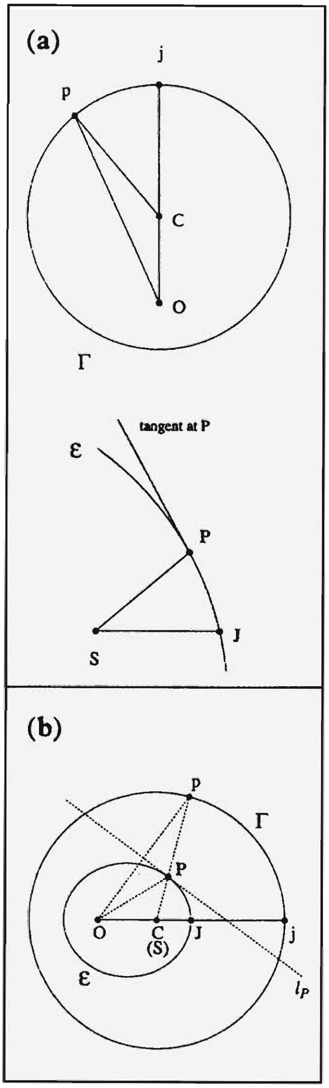
Viewing the system at instants of time such that $\vec{r}$ advances around the orbit through equal angles $\Delta\theta$, we see from equation (8) that the polygon whose vertices are the tips of the v's has equal sides (each side measures $(k/2h)\ \Delta\theta$) and equal angles (each exterior angle measures $\Delta\theta$). This means that the polygon is regular! In the limit as $\Delta\theta \to 0$, the $\vec{v}$'s span a circle; that is, the particle describes a circle in velocity space, say with centre $\vec{c}$ (and diameter $w = k/h$).

Feynman shows next that *as the particle orbits around S, the vectors $\vec{v} - \vec{c}$ and $\vec{r}$ stay at right angles to one another.* For the proof, let $\vec{r}_X$ and $\vec{v}_X$ respectively denote the position and velocity vectors corresponding to an arbitrary point X on the orbit. Let $A, B, C, \ldots$ be n points on the orbit corresponding to advances by $\vec{r}$ around the orbit through equal angles of $360°/n$. Then, as per the argument above, in velocity space the points $\vec{v}_A, \vec{v}_B, \vec{v}_C, \ldots$ are the vertices of a regular polygon whose sides subtend angles of $360°/n$ at its centre $\vec{c}$. The equality of angles shows that *as the planet moves in its orbit, the position vector $\vec{r}$ turns at the same rate as $\vec{v} - \vec{c}$.* Let J be the point of closest approach to the sun; then $\vec{v}_J$ is perpendicular to $\vec{r}_J$. (Else J would not be the point of closest approach). By the law of areas, the velocity $\vec{v}_J$ at J has the largest magnitude amongst all the $\vec{v}$'s and therefore passes through $\vec{c}$ (remember that the origin of the $\vec{v}$'s is some off-centre point); so $\vec{v}_J - \vec{c}$ is perpendicular to $\vec{r}_J$. The equality of rates of turn now implies that $\vec{v} - \vec{c}$ is perpendicular to $\vec{r}$ at each point on the orbit.

Our task therefore reduces to finding a curve $\mathcal{E}$ with the following property (see *Figure 5A*). Given: (a) a circle Γ with centre C, an off-centre point O which represents the origin, and the point j where the ray OC meets Γ, (b) a fixed point S, and a point J on $\mathcal{E}$ such that SJ is perpendicular to Oj. Then $\mathcal{E}$ must be such that for each point P on $\mathcal{E}$, if p is the point on Γ such that angle pCj equals angle PSJ, then the tangent at P to $\mathcal{E}$ is parallel to Op.

If we interchange the phrases 'perpendicular to' and 'parallel to', we arrive at an equivalent problem which luckily for us is easier to solve—because we already know the so-

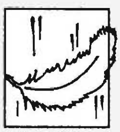
Figure 5. Feynman's proof of the ellipse theorem.



lution! *Figure 5b* tells us how to proceed: we juxtapose S upon C; rotate the velocity diagram through 90° in a clockwise direction so that J lies on Oj; and then scale the orbit diagram so that J lies at the mid-point of Oj. The requirement now is the following: $\mathcal{E}$ must pass through J, and if segment Cp intersects $\mathcal{E}$ at P, then the tangent to $\mathcal{E}$ at P must be perpendicular to Op. Lemma 4(b) implies, as a corollary, that the ellipse with foci at O and C, with major axis equal to the radius of the circle, fits the requirements exactly; for the tangent in question is just the perpendicular bisector of Op. Topping off this observation with a uniqueness argument, we see that an ellipse (or more generally, a conic section) is the answer we seek. (The proof has been given a bit sketchily, but the reader should be able to fill in the details. See T Padmanabhan in Suggested Reading for a related but somewhat different treatment.)

Contrasting the two proofs, we see that Newton's proof has great power and invites application to different force laws, whereas Feynman's proof, though elegant and simple (the geometer in Newton would have rejoiced at its transparent simplicity!), is 'custom built'; it exploits very finely certain properties possessed only by conics, but as a consequence is rather limited; for instance, one would be hard put to adapt the proof to a different force law. However this need not detract from its intrinsic worth. There are many proofs in mathematics which are custom built in this sense and which nevertheless are marvels of ingenuity.

As a final comment, we note that the Maxwell-Feynman approach yields another form of Kepler's third law which holds for open orbits as well as closed orbits: *The product of the radius of the velocity circle and the rate at which area is swept out by the radius vector has the same value for all orbits.* (See A Lenard in Suggested Reading for details.)



'Tomorrow is going to be wonderful, because tonight I do not understand anything'

Niels Bohr

Suggested Reading

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