Chaos

1. Introduction to Chaos

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Chaos is found in many different situations from epidemics like measles to fluctuations in the stock market. In this part we describe the essential features of chaos.

Introduction

The solution to most physical problems involves setting up and solving differential equations. For example, in mechanical systems, Newton's laws provide us with the required second-order differential equations whose solution gives the path taken by the system being studied. As most systems are non-linear, until recently approximations had to be used for solving these equations analytically. On solving some non-linear equations using these techniques, it is found that the dynamics is very different from that of linear equations, one notable feature being the tendency of very close initial conditions to lead to completely different motions. This is the central identifying feature of *chaotic* systems. Such systems therefore appear unpredictable, since small changes in initial conditions can become amplified.

A Simple Dynamical System

As an example, consider the equation of motion for a damped and driven pendulum, a physical realisation of which, say, is an adult pushing a child on a swing (see *Figure 1*),

$$\frac{d^2x}{dt^2} = -k\sin x - b\dot{x} + E\sin\omega t \qquad (1)$$

We take the constants b, k and E to be positive. This problem requires initial values (such as $x(0)$ and $\dot{x}(0)$), and its solution gives the motion of the system (x is the angle from the vertical) as a function of time, t. Here the non-linear term is the first on the right hand side, the damping term is the second on the right hand side, proportional to the velocity $\dot{x}$, and the driving is represented by the last term.

Figure 1. Child on a swing.

The equations of motion for such dynamical systems can have several types of solutions depending on the initial conditions or other parameters in the problem. In this particular case of the oscillator, if the damping is strong compared to driving, the motion will stop after sometime, i.e. the solution is a constant. If the driving is of the order of damping, some sort of balance is achieved and the position starts oscillating. For example, if the damping strength $b=0.2$, and the driving $E=0.5$, $x(t)$ is oscillatory. Furthermore, the motion also repeats after some time; such a behaviour is termed periodic. In the above two examples, it is also observed that any initial condition leads to the same final motion. However, for certain values of the driving constant, for example $E=0.6$, the time period becomes infinite, that is to say the motion never repeats itself even though it is still oscillatory, and different initial conditions give very different motions, and the motion is said to be chaotic. The time series in *Figures 2* and *3* illustrate the above.

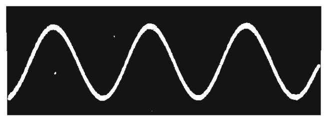


Figure 2. Time series of a pendulum with E = 0.5 (periodic).

Trajectories and Orbits

All the variables needed to completely describe a system specifies the *phase space*. The state of the system, which is a point in the phase space, can change in time, and the path so traced is known as the *trajectory* of the system. Each trajectory is uniquely specified by the initial conditions (*Box 1*).

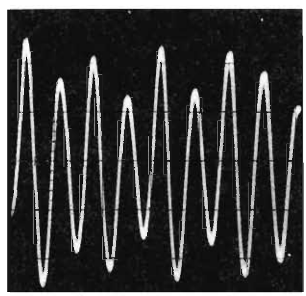


Figure 3. Time series of a pendulum with E = 0.7 (chaotic).

It is well-known that any system of differential equations may be written as a coupled set of n first order equations,

$$\dot{x}_i = f_i\,(x_1, x_2, \ldots, x_n, t), \;\; i = 1, 2, \ldots, n. \qquad (2)$$

Box 1

To specify the position of a single particle, we need its Cartesian coordinates *x, y, z*. Another set of three numbers is required in order to fix the corresponding components of its velocity. Thus for *N* particles we need 6*N* independent quantities. The state of the system can then be represented as a point in a 6*N*-dimensional abstract space called *phase space*. The motion of the whole system then corresponds to the trajectory of this single representative phase point in this phase space. Deterministic dynamics implies that there is a unique trajectory through any given phase or state point, and it is calculable in principle. This absolute determinism was probably first recognised by the 19th century French mathematician Pierre Simon de Laplace. But the Laplacian determinism, is now known to have serious errors. The macroscopic uncertainty, or rather the unpredictability, emerges in an entirely different and rather subtle manner out of the very deterministic nature of the classical laws. When this happens we say that we have *chaos*.

Chaos occurs in many places. It is in the ECG traces of patients with arrhythmiac hearts and epidemics like measles. It lurks in the fluctuations of stock prices. It is seen in the populations of species competing for limited resources in a given region. The irregular pattern of reversals of the earth's magnetic field is suspected to be due to the chaotic geodynamo. Chaos is implicated in the orbits of stars around the galactic center. The list is endless.

The variables x_i are the phase space coordinates, and the functions f_i are specified by the equations of motion.

In order to find out the phase space coordinates of the damped, driven, (non-linear pendulum) described by equation (1), we decompose it into two first order equations, writing x as x_1 and $\dot{x}$ as x_2 respectively, as

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -bx_2 - k\sin x_1 + E\sin\omega t\end{aligned} \qquad (3)$$

from which it is clear that phase space variables are x and $\dot{x}$. The trajectories of the pendulum during periodic and chaotic behaviours are shown in *Figures 4 and 5*.

There are two broad categories of dynamical systems – dissipative and conservative – based on whether the dynamics conserves

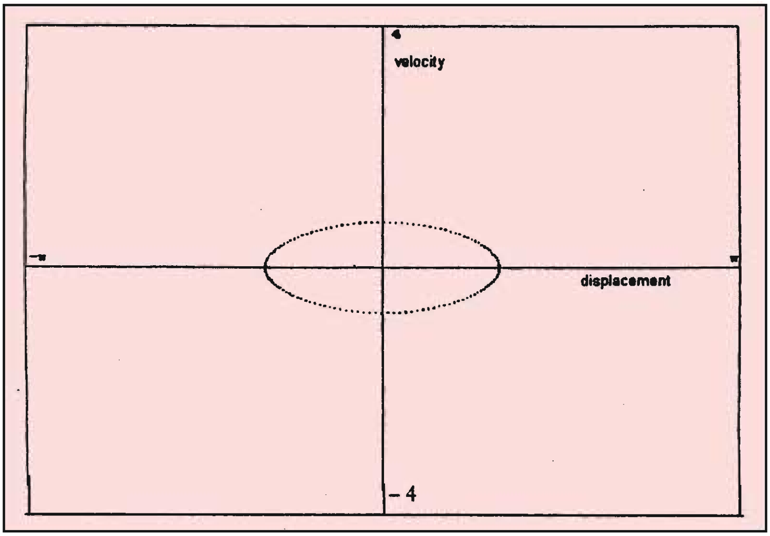


Figure 4. Phase space portrait of a periodic pendulum (limit cycle).

phase space volumes or not. In the present article, we shall consider dissipative systems, namely those where phase space volumes are not conserved as a function of time. Most common examples of dynamical systems where friction or damping plays a role are dissipative. In equation (1), for example, the term $b\dot{x}$ implies that energy is being taken out of (or dissipated from) the system.

The more general feature of dissipative systems is the shrinking of phase space volumes. If we evolve initial conditions lying on

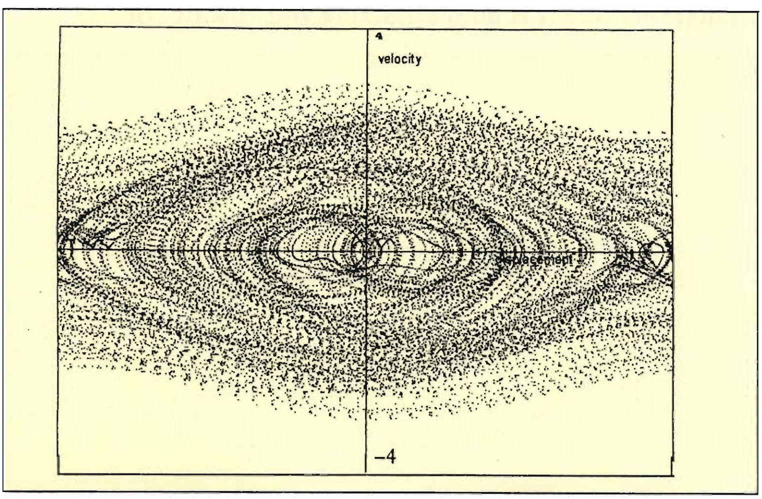


Figure 5. Phase space portrait of a chaotic pendulum.

The *period* of an orbit is defined as the time taken by the trajectory to come back to a point on the orbit in phase space.

a surface enclosing a volume V_0, the volume enclosed by the surface described by these points at time t, $V_t < V_0$. This happens if $\nabla \cdot \mathbf{f} < 0$ everywhere, where $\mathbf{f}$ is the vector field $(f_1, f_2, f_3, \ldots, f_n)^T$ (where T stands for transpose) in equation (2). This property of phase space contraction leads to the existence of 'attractors', namely those regions of phase space which are the limiting sets V_t as $t \to \infty$. The *basin of attraction* is the set of initial conditions that will eventually land up on the attractor.

Considering again the damped and driven pendulum, if $b=0.2$ and $E<0.4$, the trajectory always spirals to the origin regardless of how the pendulum is started off (*Figure 6*): the attractor is the point ($x=0$, $\dot{x}=0$). It is also called a *fixed point*. If $E=0.4$, the trajectory will converge to an ellipse (*Figure 4*), irrespective of the initial condition. This attracting ellipse, on which the motion is periodic, is usually termed a *limit cycle*. The *period* of an orbit is defined as the time taken by the trajectory to come back to a point on the orbit in phase space. In the above example, the time period, $2\pi/\omega$, is the time taken to complete a cycle around the ellipse described by equation (3).

But if a system is bounded in phase space and the time period is infinite, the system is aperiodic or chaotic; the trajectory must never close. Since the phase space volume keeps shrinking, the motion must remain bounded, and such an attractor is termed 'strange' because it is both attracting and chaotic (*Box 2*).

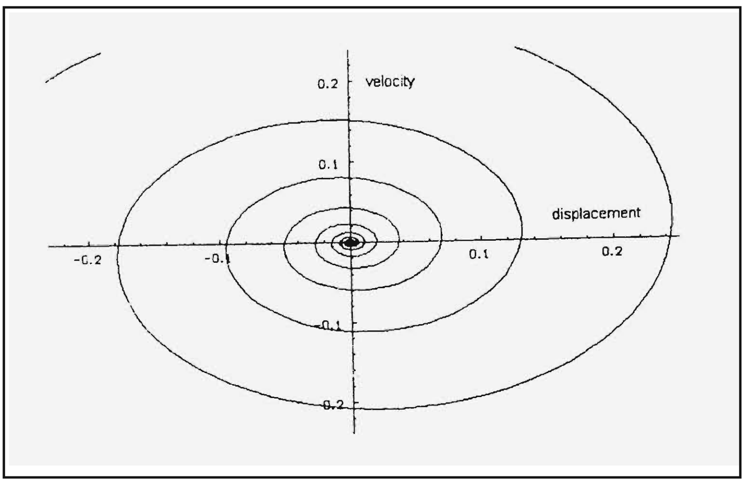
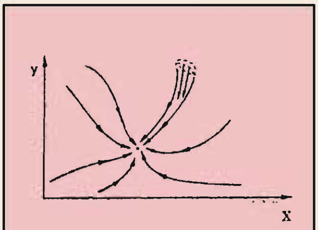


Figure 6. Phase space portrait of a limit point (damped pendulum).

Box 2

Attractors are geometric forms in the phase space to which the phase trajectories of the dynamical system converge, or are attracted and on which they eventually settle down. This happens independently of the initial conditions.

The simplest attractor is a *fixed point*, or rather a *stable fixed point*. Consider the phase portrait of a damped linear oscillator, a pendulum with friction for example. The phase space is 2-dimensional comprising the velocity (momentum) and the position coordinates. Because of damping, the phase point spirals in onto the origin and rests there (see *Figure 6*).

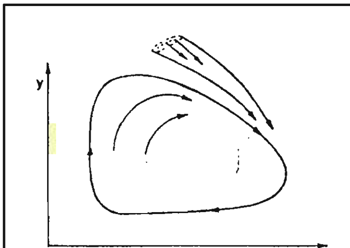


A. Stable fixed point

Also, all trajectories, no matter where they begin, are attracted towards it (see *A*) hence the name attractor. It is also readily seen, that any element of phase space will contract in its extension as it is attracted towards the fixed point. As there is contraction along both the independent directions in the 2-dimensional phase space, both the Lyapunov exponents are negative. This 'contraction' is a consequence of 'dissipative flow'. Most interesting flows in nature are dissipative and have to be maintained by external driving forces, such as stirring, heating, pumping, kicking, etc.

The *limit cycle* is a closed loop in the phase space to which the trajectories eventually converge (see *B*). The limit cycle corresponds to a stable oscillation. Here one Lyapunov exponent is zero (along the loop) and the rest are negative. Again the flow is dissipative.

There is an entirely different type of attractor that characterises flows which are chaotic, that is unstable and unpredictable in the long term. Such an attractor is a phase space non-filling set of points or orbits to which all trajectories from the outside converge, but on which neighbouring trajectories diverge. Thus atleast one Lyapunov exponent has to be positive. It is strange in respect of the geometry of its form as well as in terms of its manner of traversal of this finite region of phase space. It is called a *strange attractor*. It was discovered jointly by the Belgian mathematician David Ruelle and the Dutch mathematician Floris Takens around 1971. *Strange attractor* makes a trajectory remain confined forever to a finite region of phase space without intersecting itself. If the system happens to be dissipative, eventually the phase-space volume must contract to zero. The conflict of demands for zero volume and eternal self-avoidance is resolved by making the attractor into a *fractal* (see Classroom section of *Resonance*, Vol.2, No.10,1997). Fractal is an ingenious way of having surface without volume.



B. Limit cycle

If the exponent λ is positive, then the trajectories diverge from each other rapidly. Such sensitivity to initial conditions is termed *chaos*.

This brings us to an interesting point about such systems. If we consider two dimensional systems, two types of attractors are possible – a limit point or a limit cycle. It can be seen intuitively that for a trajectory to remain bounded and never close in a plane, it must intersect, which is not possible as differential equations have unique solutions. Therefore it follows that chaotic or strange attractors are possible only in systems with more than two dimensions. The case of the driven damped pendulum system, equation (1) is one with more than two degrees of freedom since one must specify the forcing also to completely determine the motion, and the three coordinates of the 3-D phase space are ($x_1 \equiv x$, $x_2 \equiv \dot{x}$, $E \sin\omega t$).

The defining characteristic of chaotic systems is that if one evolves two trajectories from different initial conditions, they look completely different after some time, no matter how close the initial conditions were. This feature is characterised by an exponential rate of separation of the two trajectories, and this divergence is measured by the Lyapunov exponent λ, which is defined through the relation,

$$d_t = d_0 \exp(\lambda t), \tag{4}$$

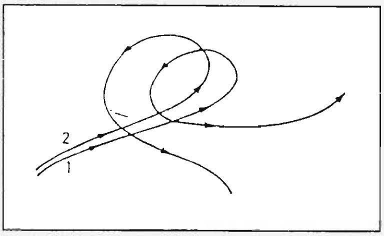
or equivalently, by

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \lim_{d_0\to 0} \log \frac{d_t}{d_0}, \tag{5}$$

where d_0 is the initial separation of two points in the phase space of the system, d_t is the separation after time t of two trajectories started off from those two points. If the exponent λ is positive, then the trajectories diverge from each other rapidly. Such sensitivity to initial conditions is termed *chaos*. Calculation of the Lyapunov exponent is often the most reliable method of determining whether a system is chaotic or not (*Box 3*).

Box 3

How can a system be deterministic and yet become chaotic? The answer lies in the sensitive dependence on initial conditions. The deterministic laws permit a given initial state to evolve to a unique and calculable state of the system at any future instant of time. What if the initial conditions are known only approximately? If we start identical systems from two neighbouring state points, then we generally expect their trajectories to stay close by for *all* future times. Such a system is said to be well behaved. What then if the initial errors actually grow with time, say exponentially? In our phase space picture then, any two trajectories that started off at some neighbouring points initially, will begin to diverge so that the line joining them will get stretched exponentially ie. like $e^{\lambda t}$, with time (see *A*). Here λ measures the rapidity of divergence or convergence accordingly as it is positive or negative. It is called the *Lyapunov exponent*. This is precisely what is meant by the sensitive dependence on initial conditions. It makes the flow in the phase space complex, almost random. For, then the approximately known initial conditions do not give the distant future states with comparable approximation. This is often referred to picturesquely as the *Butterfly Effect*. The flapping of a butterfly's wings in Brazil may set off a tornado in Texas.



A. Butterfly effect.

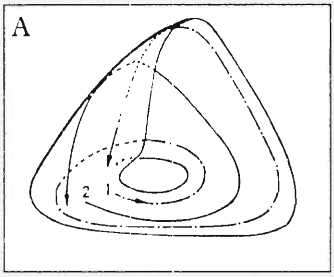
It is however important to note that when measuring the Lyapunov exponent, one ensures that the system has evolved for a sufficiently long time (*the transient period*), for its trajectory to lie on the attractor. Measuring the Lyapunov exponent is of significance only at the attractor. Incidentally, for a given positive Lyapunov exponent if two trajectories come close, equation (4) holds and the trajectories must subsequently diverge (*Box 4*).

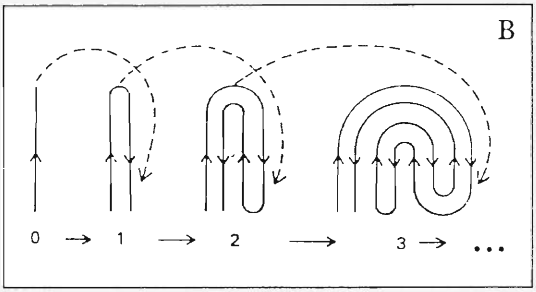
Box 4

There are a number of computer models with a small number of degrees of freedom that show chaos. The question is what makes them chaotic and the answer lies in non-linearity. It is non-linearity that makes the system sensitive to initial conditions. It can amplify small changes. A dynamical system with a small number of degrees of freedom will have a low dimensional state space. Moreover, the degrees of freedom are expected to have a finite range. For example, we do not expect the velocities to be infinite. Thus the phase point will be confined to a finite region of the low dimensional state space. This raises a geometrical question of packing. How does the phase trajectory, confined to a finite region of

Box 4 continued...

the low dimensional state space, wind around forever without intersecting itself or closing on itself? We know that two trajectories from neighbouring points diverge, stretching the line joining them exponentially. But this cannot go on indefinitely because of the finiteness of the range. The trajectories must fold back, and may approach each other only to diverge again (see *A*). This process is analogous to a baker's way of handling dough. He rolls the dough to stretch it out and then folds it, and then repeats the process again and again. If we have a coloured spot on the dough, the spot would have spread out throughout the dough colouring it apparently uniformly, suggesting thorough mixing. But actually it is finely structured. The neighbouring points on the spot would have diverged and become totally uncorrelated after a few rounds of baker's transformation (see *B*). Stretching with folding is a highly non-linear process that generates the above sensitivity. Baker's transformation seems a general algorithm for chaotic evolution.





A. Sensitive dependence on initial conditions: divergence-cum-folding-back of neighbouring trajectories.

B. Successive stages of stretching and folding-back of baker's transformation.

So much for a description of chaos. We take up in the next part the routes to chaos.

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Suggested Reading

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