Multiplication on $\mathbb{R}^n$

2. Adam's Theorems and Applications

Basudeb Datta

Basudeb Datta received his Ph D degree from the Indian Statistical Institute in 1988. Since July 1992 he has been at the Indian Institute of Science, Bangalore. His areas of interest are Topology and Geometry.

In the first part of this article we discussed the non-existence of division algebra structures and vector products on $\mathbb{R}^n$. We now show how results from *topology* can be used to prove the results in the first article. While the discussion in the first part of the article required only elementary mathematics, this (second) part requires notions from basic *topology*, with which, we hope, many students at the M Sc level are familiar.

Continuous Multiplication

In Part 1, we discussed the division algebra structures on $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^4$ and $\mathbb{R}^8$ In each case if we restrict our multiplication to $\mathbb{R}^n \setminus \{0\}$ (n = 1, 2, 4, or 8 respectively) then we get a continuous mapping

$$m : (\mathbb{R}^n \setminus \{0\}) \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}^n \setminus \{0\}$$

with the element e (= 1, $(1,0)$, $(1,0,0,0,)$ or $(1,0,0. \quad ,0)$ respectively) in $\mathbb{R}^n \setminus \{0\}$ such that $m(e,x) = m(x,e) = x$ for all $x \in \mathbb{R}^n \setminus \{0\}$. The following theorem is about the converse.

Theorem T: *If there exists a continuous map* $m : (\mathbb{R}^n \setminus \{0\}) \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}^n \setminus \{0\}$ *with an element* $e \in \mathbb{R}^n \setminus \{0\}$ *such that* $m(e,x) = m(x,e) = x$ *for all* $x \in \mathbb{R}^n \setminus \{0\}$ *then* $n = 1, 2, 4$ *or 8.*

This is a theorem in topology and follows from a more general theorem (Theorem $*$ below) of topology proved by J F Adams in 1962, and by J F Adams and M F Atiyah in 1966, using sophisticated methods from topology. To state this theorem we need to define what is called a Hopf space. Before that let us recall the definition of a topological group. A *topological group* is a Hausdorff topological space X with a group structure in which the group operations are continuous.[1]

A Hausdorff topological space with a groupoid[2] structure is called a *topological groupoid* if the composition operation is continuous. So, each topological group is a topological groupoid.

Clearly, $(\mathbb{R}^+, \cdot)$, $(\mathbb{C}\setminus\{0\}, \cdot)$, $(\mathbb{H}\setminus\{0\}, \cdot)$, $(\mathbb{R}^n, +)$ (n any natural number) are topological groups. If G is a finite group then G is a topological group with discrete topology. Consider the unit sphere S^n in $\mathbb{R}^{n+1}$ i.e.,

$$S^n = \{(x_1, \quad , x_{n+1}) : x_1^2 + \cdots + x_{n+1}^2 = 1\}.$$

(Topology on S^n is of course the induced topology from $\mathbb{R}^{n+1}$.) Then $(S^0, \cdot)$, $(S^1, \cdot)$ and $(S^3, \cdot)$, where the multiplication is induced from those in $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ respectively, are topological groups. Also the multiplication in $\mathbb{O}$ induces a multiplication in S^7 which satisfies all the properties of a topological group except the associative law. So, $(S^7, \cdot)$ is a topological groupoid.

A H-space is a generalisation of a topological groupoid. For more about H-spaces, see Dugundji in Suggested Reading.

H-space: A topological space X with a continuous mapping $m : X \times X \to X$ is called a Hopf space (or a H-space) if there exists an element e in X such that both $m(e,\text{-})$ and $m(\text{-},e) : X \to X$ are homotopic[3] to the identity mapping.

Clearly, each topological groupoid is an H-space. In particular, $(S^0, \cdot)$, $(S^1, \cdot)$, $(S^3, \cdot)$ and $(S^7, \cdot)$ are H-spaces. The following theorem is about the converse (see Husemoller, pp.202 in Suggested Reading).

Theorem $*$ (Adams, 1962): *The only spheres with an H-space structure are S^0, S^1, S^3 and S^7.*

[1] More precisely, a topological group is a Hausdorff topological space together with a continuous mapping $m \quad X \times X \to X$ such that (i) there exists $e \in X$ such that $m(e,x) = x = m(x,e)$ for all $x \in X$, i.e., $m(e,\text{-}) = \mathrm{Id}_X = m(\text{-},e)$ (such an e is unique and called the identity), (ii) $m(x, m(y,z)) = m(m(x,y),z)$ for all x, y, $z \in X$ (associative law), (iii) for each x in X, there exists an element (which is of course unique), denoted by x^{-1} and called the inverse of x, in X such that $m(x,x^{-1}) = e = m(x^{-1},x)$ and (iv) the mapping $l \quad X \to X$, given by $l(x) = x^{-1}$, is continuous.

[2] A set X with a composition (a map from $X \times X$ to X) is called a groupoid if there exists an identity.

[3] Let X and Y be two topological spaces. Two mappings f, $g \quad X \to Y$ are called *homotopic* if there exists an 1-parameter family of mappings $F_t \quad X \to Y$, $t \in I = [0,1]$, such that $F_0 = f$ and $F_1 = g$ and it depends continuously on t. More precisely, f and g are homotopic if there exists a continuous mapping $F \quad X \times I \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all x in X.

The proof of Theorem $*$ is beyond the scope of this article. However, we deduce other theorems from Theorem $*$ in this article.

Proof of Theorem T: Let $m : (\mathbb{R}^n \setminus \{0\}) \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}^n \setminus \{0\}$ be a multiplication with identity e. Define $\mu : S^{n-1} \times S^{n-1} \to S^{n-1}$ as $\mu(x,y) = m(x,y)/\|m(x,y)\|$. Clearly μ is continuous. Let $\varepsilon = e/\|e\|$. Consider the continuous mapping $F : S^{n-1} \times I \to S^{n-1}$ given by[4]

[4] See Part 1 of this article for equations (1) to (13).

$$F(x,t) = \frac{m(t\varepsilon + (1-t)e, x)}{\|m(t\varepsilon + (1-t)e, x)\|}. \tag{14}$$

(Observe that $t\varepsilon + (1-t)e \neq 0$ for all $t \in [0,1]$.) Then $F(x,0) = x$ and $F(x,1) = \mu(\varepsilon, x)$ for all $x \in S^{n-1}$. Therefore $\mu(\varepsilon, \text{-})$ is homotopic to the identity mapping. Similarly, one can show that $\mu(\text{-}, \varepsilon)$ is also homotopic to the identity map. Thus μ defines an H-space structure on S^{n-1}. Hence, by Theorem $*$, $n - 1 \in \{0, 1, 3, 7\}$. This proves Theorem T.

We now prove the following two theorems (which we stated in Part 1):

Theorem A: $\mathbb{R}^n$ *has a division algebra structure over* $\mathbb{R}$ *if and only if* $n \in \{1, 2, 4, 8\}$.

Theorem L: $\mathbb{R}^p$ *has a vector product if and only if* $p \in \{1, 3, 7\}$.

Proof of Theorem A: We have already observed the 'if part' in Part 1. If $\mathbb{R}^n$ has a division algebra structure, then it has no divisors of zero. The 'only if part' now follows from Theorem T.

Proof of Theorem L: We have already observed the 'if part' in Part 1. Assume $\mathbb{R}^p$ $(p \geq 0)$ has a vector product ν. Consider $\mathbb{R}^p$ as a subspace of $\mathbb{R}^{p+1}$ in the following sense:

$$\begin{aligned} \mathbb{R}^p &= \text{Span}_{\mathbb{R}}(\{e_1, \quad , e_p\}) \\ &\subseteq \text{Span}_{\mathbb{R}}(\{e_0, e_1, \quad , e_p\}) = \mathbb{R}^{p+1} \end{aligned}$$

Then any vector $\tilde{x}$ in $\mathbb{R}^{p+1}$ can be expressed uniquely as $\tilde{x} = x_0 e_0 + x$ where $x_0 \in \mathbb{R}$ and $x \perp e_0$. Define $m : \mathbb{R}^{p+1} \times \mathbb{R}^{p+1} \to \mathbb{R}^{p+1}$ as

$$m(\tilde{x}, \tilde{y}) = (x_0 y_0 - \langle x, y \rangle) e_0 + x_0 y + y_0 x + \nu(x, y). \quad (15)$$

Then clearly m is continuous and

$$m(\tilde{x}, e_0) = \tilde{x} = m(e_0, \tilde{x}), \quad \text{for all } \tilde{x} \in \mathbb{R}^{p+1} \quad (16)$$

Now, if $m(\tilde{x}, \tilde{y}) = 0$ then we get

$$x_0 y_0 = \langle x, y \rangle, \quad x_0 y + y_0 x = 0 \quad \text{and} \quad \nu(x, y) = 0. \quad (17)$$

From (4) and (17) one can see easily that $\tilde{x} = 0$ or $\tilde{y} = 0$. So, $\tilde{x} \neq 0$ and $\tilde{y} \neq 0$ imply $m(\tilde{x}, \tilde{y}) \neq 0$. Therefore we can restrict our m to $(\mathbb{R}^{p+1} \setminus \{0\}) \times (\mathbb{R}^{p+1} \setminus \{0\})$ and get a map $m : (\mathbb{R}^{p+1} \setminus \{0\}) \times (\mathbb{R}^{p+1} \setminus \{0\}) \to \mathbb{R}^{p+1} \setminus \{0\}$ with the property (16). The theorem now follows from Theorem T.

Algebra gives the existence of continuous mappings from $S^n \times S^n$ to S^n for some n and Topology shows the non-existence of division algebra structure on R^n for almost all n.

Remark 1 : Quaternion multiplication is not commutative. It also follows from a theorem in topology that $\mathbb{R}^4$ does not allow any commutative division algebra structure over $\mathbb{R}$.

Remark 2 : Octonion multiplication is non-associative.Again using another theorem in topology, it can be proved that $\mathbb{R}^8$ does not allow any associative division algebra structure over $\mathbb{R}$.

Vector Fields on S^m

For $n = 2^{4\alpha+\beta}(2\gamma + 1)$, where $0 \leq \beta \leq 3$, let $k(n) := 8\alpha + 2^\beta$. In Part 1, we have stated the following theorem due to Hurwitz.

Theorem N : *If there exists a bilinear map* $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ *such that* $\|f(y, x)\| = \|y\| \cdot \|x\|$ *for all* $y \in \mathbb{R}^m$ *and* $x \in \mathbb{R}^n$ *then* $m \leq k(n)$.

Theorem N also follows from another theorem of Adams. A continuous map $V : S^{n-1} \to \mathbb{R}^n$ is called a *vector field* on

[5] Consider the unit sphere S^r in $\mathbb{R}^{r+1}$ For $x \in S^r$ let

$$T_xS^r := \{v \in \mathbb{R}^{r+1} : \langle x, v\rangle = 0\} = (\mathrm{Span}_{\mathbf{R}}(x))^{\perp}.$$

Then T_xS^r called the *tangent space* of S^r at x, is a r dimensional subspace of $\mathbb{R}^{r+1}$

For two subspaces X and Y in $\mathbb{R}^n$, we write $Y = X^{\perp}$ if $X + Y = \mathbb{R}^n$ and $x \perp y$ for all $x \in X$ and $y \in Y$. In that case we also write $\mathbb{R}^n = X \oplus^{\perp} Y$.

S^{n-1} if $V(x) \in T_xS^{n-1}$ (see margin note 5), i.e., $\langle V(x), x\rangle = 0$ for all $x \in S^{n-1}$. Vector fields $V_1, \quad , V_s$ are called *nowhere dependent* if for each $x \in S^{n-1}$, $\{V_1(x), \quad V_s(x)\}$ is a set of independent vectors in $\mathbb{R}^n$ By using Hurwitz's map $f : \mathbb{R}^{k(n)} \times \mathbb{R}^n \to \mathbb{R}^n$ one can easily construct (see Husemoller. pp.140, in Suggested Reading) $k(n) - 1$ nowhere dependent vector fields on S^{n-1}. In 1962, J. F. Adams proved (see Husemoller, pp.225, in Suggested Reading) the following, the proof of which is beyond the scope of this article.

Theorem V: *If S^{n-1} allows r nowhere dependent vector fields then $r \leq k(n) - 1$.*

Proof of Theorem N: Let $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ be a bilinear map such that $\|f(y, x)\| = \|y\| \; \|x\|$ for all $y \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$ Let $\{e_1, \quad , e_m\}$ denote the standard basis of $\mathbb{R}^m$ Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be the map defined by

$$g(x) = f(e_m, x). \tag{18}$$

Then g is linear and

$$\|g(x)\| = \|x\| \quad \text{for all } x \in \mathbb{R}^n \tag{19}$$

This implies g is non-singular. For $x \in S^{n-1} \subseteq \mathbb{R}^n$ let $h_x : \mathbb{R}^m \to \mathbb{R}^n$ be the map defined by

$$h_x(y) = f(y, g^{-1}(x)). \tag{20}$$

Then by (19) we have:

$$\begin{aligned}\|h_x(y)\| &= \|f(y, g^{-1}(x))\| = \|y\|\|g^{-1}(x)\| \\ &= \|y\|\|x\| = \|y\| \;\; \text{for all } y \in \mathbb{R}^m \end{aligned} \tag{21}$$

and so, by replacing y by $y + z$ in (21), we get:

$$\langle h_x(y), h_x(z)\rangle = \langle y, z\rangle \quad \text{for all } \; y, z \in \mathbb{R}^m \tag{22}$$

Therefore, $\{h_x(e_1), \quad , h_x(e_m) = x\}$ is a set of pairwise orthogonal vectors in $\mathbb{R}^n$ Now, for $1 \leq i \leq m$, define

$$X_i(x) = h_x(e_i), \quad \text{for } \; x \in S^{n-1}. \tag{23}$$

Then $X_i(x) = f(e_i, g^{-1}(x))$ and hence X_i is continuous for each $i \in \{1, \ldots, m\}$. Since, $h_x(e_1), \ldots, h_x(e_m)$ are mutually orthogonal, $X_1, \ldots, X_{m-1}$ are $m-1$ nowhere dependent vector fields on S^{n-1} and so, by Theorem V, $m - 1 \leq k(n) - 1$. This proves the theorem N.

Complex Geometry

We would like to discuss one more application of Theorem $*$. This time in geometry. For that we need the definition of *almost complex structure*. The definition of almost complex structure on general manifolds is beyond the scope of this article. We will define almost complex structure on spheres, since that is all that is required here.

Let V be a vector space over $\mathbb{R}$. If there exists a linear mapping $J : V \to V$ such that $J \circ J = -$ Id, then J is called a complex structure on V In that case, $iv := J(v)$ gives a complex vector space structure on V If V is finite dimensional then clearly V is even dimensional.

> **Almost complex structure on spheres:** A family $J = \{J_x : T_xS^r \to T_xS^r,\ x \in S^r\}$ of linear maps is called an almost complex structure on S^r if (i) $J_x \circ J_x = -$ identity mapping on T_xS^r for all $x \in S^r$ and (ii) the map $f : S^r \times \mathbb{R}^{r+1} \to \mathbb{R}^{r+1}$, given by $f(v, u) = J_v(u - \langle v, u\rangle v)$, is continuous.

In that case, J_x varies continuously with x and for each $x \in S^r$ J_x defines a complex vector space structure on T_xS^r

Let $\mathbb{R}^7$ (respectively $\mathbb{R}^3$) be the imaginary subspace of $\mathbb{O}$ (respectively $\mathbb{H}$). Consider the unit sphere S^6 (respectively S^2) in this $\mathbb{R}^7$ (respectively in $\mathbb{R}^3$). Consider the class J of mappings as follows:

$$J_v : T_vS^6 \to T_vS^6 = (\mathrm{Span}_{\mathrm{R}}(v))^{\perp} \subseteq \mathbb{R}^7$$

$$J_v(u) = v \times u = v \cdot u \qquad (24)$$

S^2 and S^6 have almost complex structures.

(same for S^2 also). By using (7), (9) and Observation 1, we get $J_v \circ J_v = .-\mathrm{Id}$. So, S^6 and S^2 have almost complex structures. In 1953, A. Borel and J. P. Serre proved the following.

Theorem G : *S^{2n} allows an almost complex structure if and only if $2n = 2$ or 6.*

We present here a proof of Theorem G using Theorem $*$.

Proof: We have already proved the 'if part'.

Conversely, assume S^{2n} has an almost complex structure J. We also assume $n > 0$! Consider S^{2n} in $\mathbb{R}^{2n+2}$ in the following sense:

$$\begin{aligned} S^{2n} \subseteq \mathbb{R}^{2n+1} &= \mathrm{Span}_{\mathbf{R}}(\{e_1, \quad , e_{2n+1}\}) \\ &\subseteq \mathrm{Span}_{\mathbf{R}}(\{e_0, e_1, \quad , e_{2n+1}\}) = \mathbb{R}^{2n+2} \end{aligned}$$

Then for each $x \in S^{2n}$ we have

$$\mathbb{R}^{2n+2} = \mathrm{Span}_{\mathbf{R}}(\{e_0, x\}) \oplus^{\perp} T_x S^{2n}. \tag{25}$$

If $\tilde{x} \in S^{2n+1}$ (the unit sphere in $\mathbb{R}^{2n+2}$), then either $\tilde{x} = \pm e_0$ or there exists unique $x \in S^{2n}$ such that

$$\tilde{x} = ae_0 + bx, \quad \text{where } b > 0. \tag{26}$$

Consider the mappings $\mu = S^{2n} \times S^{2n} \to \mathbb{R}^{2n+2}$ given by

$$\mu(x, y) = -\langle x, y\rangle e_0 + J_x(y - \langle x, y\rangle x), \tag{27}$$

and $\tilde{\mu} : S^{2n+1} \times S^{2n+1} \to \mathbb{R}^{2n+2}$ given by

$$\begin{aligned} \tilde{\mu}(e_0, \tilde{y}) &= \tilde{y}, \quad \tilde{\mu}(-e_0, \tilde{y}) = -\tilde{y} \quad \text{and} \\ \tilde{\mu}(ae_0 + bx, ce_0 + dy) &= ace_0 + bcx + day + db\mu(x, y) \\ &= (ac - db\langle x, y\rangle)e_0 + (bc + \langle x, y\rangle da)x + \\ &\quad da(y - \langle x, y\rangle x) + dbJ_x(y - \langle x, y\rangle x), \end{aligned} \tag{28}$$

when $b > 0$. First observe that $\mu(x, y) = 0$ implies $\langle x, y\rangle = 0$ and $J_x(y - \langle x, y\rangle x) = 0$. These imply $y = 0$, a contradiction.

So, $\mu(x,y) \neq 0$ for all $x, y \in S^{2n}$. Clearly, μ is continuous and it is not difficult to check that $\tilde{\mu}$ is also continuous.

Claim : $\tilde{\mu}(\tilde{x},\tilde{y}) \neq 0$ for all $\tilde{x},\ \tilde{y} \in S^{2n+1}$.

If $\tilde{x} = \pm e_0$ or $\tilde{y} = \pm e_0$ then the claim is trivially true. So, let us assume $\tilde{\mu}(\tilde{x},\tilde{y}) = 0$, where $\tilde{x} = ae_0 + bx$, $\tilde{y} = ce_0 + dy$ and $b,\ d > 0$. Then we get

$$ac - \langle x,y\rangle db = 0 = bc + \langle x,y\rangle da \quad \text{and} \tag{29}$$

$$a(y - \langle x,y\rangle x) + bJ_x(y - \langle x,y\rangle x) = 0. \tag{30}$$

Applying J_x on both sides of (30) we get

$$aJ_x(y - \langle x,y\rangle x) - b(y - \langle x,y\rangle x) = 0. \tag{31}$$

From (30) and (31) we get $(a^2 + b^2)(y - \langle x,y\rangle x) = 0$ which implies $y = \pm x$. If $y = x$ then from (29) we get $ca - db = 0 = da + cb$. Therefore, (x,y) is a non-trivial solution of the system of linear homogeneous equations

$$cx - dy = 0 = dx + cy, \qquad \text{where } c^2 + d^2 \neq 0.$$

This is a contradiction. Similarly one gets a contradiction for $y = -x$ also. This proves the claim. Therefore we can define $m = S^{2n+1} \times S^{2n+1} \rightarrow S^{2n+1}$ as

$$m(\tilde{x},\tilde{y}) = \frac{\tilde{\mu}(\tilde{x},\tilde{y})}{\|\tilde{\mu}(\tilde{x},\tilde{y})\|}. \tag{32}$$

Then m is continuous and $m(e_0,\text{-}) = m(\text{-},e_0) =$ the identity map on S^{2n+1}. Thus, S^{2n+1} has an H-space structure. Therefore, by Theorem $*$, we get $2n + 1 = 1, 3$ or 7. This proves Theorem G.

Suggested Reading

- J F Adams. Vector Fields on Spheres. *Bull. Am. Math.Soc.* 68. pp. 39-41. 1962. and *Ann. Math.* 75. 603-632, 1962.
- D Husemoller. *Fibre Bundles.* Springer-Verlag. 1966.
- J Dugundji. *Topology.* Prentice-Hall of India Pvt. Ltd., 1975.

Address for Correspondence
Basudeb Datta
Department of Mathematics
Indian Institute of Science
Bangalore 560 012, India.
dattab@math.iisc.ernet.in

Nature, and Nature's laws, lay hid in night
God said, Let Newton be, and all was light.

Alexander Pope