An Improved Bound on k-sets

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Tamal K Dey (Computer Science and Engineering, Indian Institute of Technology, Kharagpur) recently made spectacular progress on an important and fundamental problem in combinatorial geometry. The problem, originally posed by the legendary Paul Erdos, is to find the least upper bound on the number of ways n points on the plane can be linearly separated into k and $n - k$ point sets. This article explains the problem and the result. It is based on the manuscript by Tamal K Dey entitled 'Improved bounds for k-sets and k levels' which has been invited to the special issue of journal *Discrete and Computational Geometry* dedicated to the memory of Paul Erdos which will appear in January 1998.

Editor

I found the professor reading something rather intently. As I came in, the professor looked up, and said "Here is something new. It is a result by Tamal K Dey from IIT Kharagpur, and it is about improved bounds on k-sets (see Tamal K Dey in Suggested Reading)."

"And" I asked "What is a k-set?"

"Suppose you put n random points on a piece of paper. Now you want to draw a line on that paper such that exactly k points lie on one side of the line. Obviously, k is smaller than n. Each such collection of k points is a k-set."

My face must have turned blank, for the professor went to the board and drew something like *Figure 1*, and asked "In how many ways can you divide these points with a line such that there are three points on one side?"

"Four."

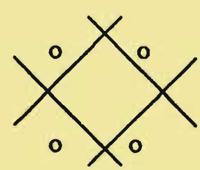
"Correct. But it is easy because we are choosing only three points on one side and one on the other. Finding this answer will be difficult if you have many more points and the value of k is not so simple."

I was still looking at *Figure 1*. "Won't the number of k sets be the same as the number of $n - k$ sets? When I draw a line to separate k points then of course I also separate $n - k$ points on the other side!"

"Good point. That's why it is sufficient to give a bound for k between 1 and the integer less than or equal to $n/2$. Now tell me: can you think of a number that will always be at least as large as the actual number of k sets given an initial set of n points?"

I thought hard, and said "What about $\binom{n}{k}$?"

Figure 1.



"Exactly, but it's too big. That's where Tamal's work comes in. People knew from 1971 that a bound is given by $O(n\sqrt{k})$, but no one could improve on that. Then Tamal..." "Just a minute. What's this $O(xyz)$?"

"It means that the number can't grow 'faster' than $n\sqrt{k}$. It can be at most a constant times $n\sqrt{k}$ but not more than that."

"Oh, so any polynomial larger than $n\sqrt{k}$ will be larger by an order of magnitude than the number, right?"

"Yes. So, Tamal improved on that and showed that it is at most $O(nk^{1/3})$. We don't know whether this is the final solution or this can be improved further, but this is the best so far."

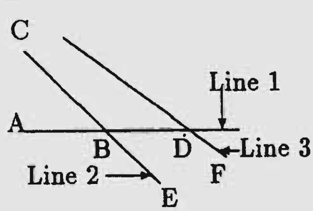
"And how does he do that?"

"To understand his proof, we have to understand a few definitions first. Tamal actually proved the result based on those shown earlier by others. He put everything in a nice framework and finally got it. So we have to understand some basics."

"OK, let's try" I said.

The professor was visibly pleased. "Fine, let's begin. First, suppose there are several lines lying in a plane like this (*Figure 2*).

Figure 2.



If you look at it, there are several levels there. For example, the part AB has no other line below it, so it is a part of the 0-level. Similarly BE. But CB and BD are parts of the 1-level as there is a line below each of them."

"There is nothing below B", I protested.

"Right, but as you move along CB towards B, every point belongs to the 1-st level, so we take B also."

"OK."

"Thus we have all levels, 0, 1, 2 .. Also, whenever two lines intersect, they give rise to a vertex, like B and D. There are levels of vertices also ; for example, B is at 0-level and D at 1-level."

"Right. Just similar to lines."

"Now suppose we start moving on line 1 from A towards B. As we reach B, we reach the 1-level of lines, at a 0-level vertex."

"Right."

"If you keep on moving, you reach D; and that's a 2-level line and a 1-level vertex!"

"Do you mean that moving on lines will always reach the k-level of line at a $(k-1)$-level vertex?"

"Exactly!! Now after reaching the k-level, if you immediately turn right, and keep on moving, turning right every time you hit a $(k-1)$-level vertex, you get a chain that is concave, i.e. bent at the top and going down on both sides. For exam-

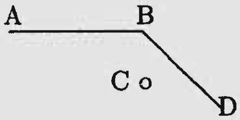


Figure 3.

ple, the line segment which contains AB and the line segment containing BE combined together form a concave chain."

"Right."

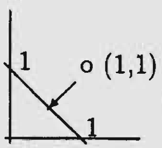
"So now let's see how Tamal proved it. He forms a graph whose vertices are those n points, and the edges are such that if you extend an edge on both sides to infinity, i.e., make it a line instead of a line segment, then there are exactly $(k-1)$ points below it (*Figure 3*). C and D are below AB extended and A and C are below BD extended.

Now, there is a trick by which you can convert a point to a line and a line to a point maintaining their relative positions. This is duality of point and lines."

"How?"

"Well, suppose the line l is $ax + b$. Then convert it to a point l^* given by $(-a, b)$. And suppose the point p is (c, d). Then convert that to a line p^* given by $-cx + d$. (*Figure 4*)."

Figure 4.



The dual of this point is this line and vice versa

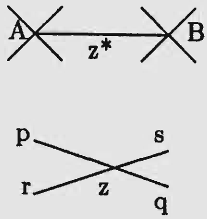


Figure 5

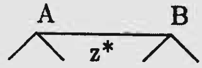


Figure 5A

"So p lies above, on or below l implies that l^* lies above, on or below p^*!"

"Correct. So in Tamal's construction, if you use this trick the edges become points and the vertices become lines. So if two edges cross in the original graph, then in the dual figure they will look like this (*Figure 5*)."

"Yes. The two edges pq and rs become the points A and B, and the lines crossing at A are actually p^* and q^*. Similarly lines crossing at B are r^* and s^*; and thus z^* is the line joining A and B."

"Beautiful. So AB is just touching the concave chain whose peak is at A as well as the concave chain whose peak is at B (*Figure 5A*). Now if you take two such concave chains, how many tangents can you draw which touch both chains?"

This time, for a change, I drew a figure (*Figure 6*). "Every tangent touches both chains. So as in *Figure 6*, suppose concave chain 1 touches both the tangents. Now no matter where concave chain 2 touches tangent 1, it has to intersect concave chain 1 at least twice to reach and touch tangent 2. Therefore each chain will definitely cut the other at least once before every touch–that means that the number of such tan-

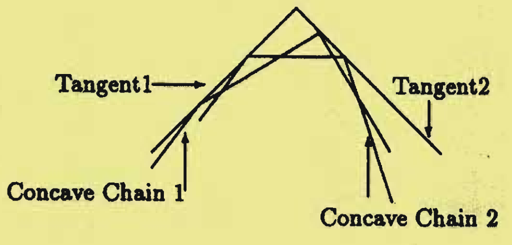


Figure 6.

gents can't be larger than the number of crossings of the chains."

"Excellent. Now in the graph, if you take all pairs of edges, then every crossing leads to one tangent in the dual as in *Figure 5*; and, the number of tangents is at most equal to the number of crossings between concave chains."

"Right."

"And, whenever concave chains cut each other, they generate a vertex."

"Yes."

"And the way we constructed the graph, the maximum level of vertices we have is $k-1$."

"Right because we started with edges such that they have exactly $(k-1)$ vertices below them, which means when we apply duality, we get points with exactly $(k-1)$ lines below them."

"So the total number of vertices, from level 0 to $k-1$, bounds the number of concave chain crossings. This sum was shown to be $O(nk)$ in 1986 (see Alon and Györi in Suggested Reading). There is a very simple argument for it. There are only k-levels in the dual graph. Therefore, there are k concave chains. Each one of the n lines can cut each one of these chains at most twice. Thus the maximum number of such intersections is at most $2nk$, which is $O(nk)$."

"But how does it lead to Tamal's result?"

"Another result says that the number of k-sets is of the order of k-level vertices (see Edelsbrunner in Suggested Reading). In the graph, as you have just said, the number of edges is equal to the number of $(k-1)$-level vertices. It was known from 1982 that in a graph, with n vertices and t edges, that is drawn on the plane, there are at least ct^3/n^2 crossings among the edges where c is a constant (see Ajtai and others in Suggested Reading). In our case t is the number of $(k-1)$-level vertices."

"Yes! So c times the number of $(k-1)$-level vertices cubed divided by n^2 is less than or equal to some constant times nk, which means that the number of $(k-1)$-level vertices is less than or equal to some constant times $nk^{1/3}$. That means for k-level vertices, the number is a constant times $n(k+1)^{1/3}$, but that is just $O(nk^{1/3})$. Hey, that's neat!"

"Parfait!! So the final result is $O(nk^{1/3})$. Actually, Tamal goes beyond this and shows other results, but I think this is the major contribution."

Suggested Reading

- Ajtai M, Chvatal V, Newborn M and Szemeredi. Crossing-free Subgraphs. *Ann. Discrete Math.* 12.pp9-12,1982.
- Alon N and Györi E. The Number of Small Semispaces of A Finite Set of Points in the Plane. *J. Combin. Theory Ser.* A41. pp.154-157, 1986.
- Edelsbrunner H. *Algorithms in Combinatorial Geometry.* Springer-Verlag. Heidelberg. Germany, 1987.
- Tamal K Dey. Improved Bounds on Planar k-sets and k-levels. to appear in the *Proc. 38th Annual Symposium on Foundations of Computer Science.* 1997. Miami, Florida; also to appear in Discrete and Computational Geometry.

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The Nobel Diploma of C V Raman