Ohm's Law, Kirchoff 's Law and the Drunkard's Walk

1. Related Electrical Networks

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Recall from Resonance, Vol. 1, No. 7 that when the Drunkard asks: "Will I ever, ever get home again?", Polya answers: "You can't miss, just keep going and stay out of 3D!" In this two-part article we show how to look at this question of recurrence and transience of random walks (which was originally asked and solved by George Polya) through electrical networks. In the first part we look at the related electrical networks.

Reticulated Resistors

Suppose n resistors are placed in a series as in *Figure 1.* Each resistor is of magnitude r ohms and between the two ends of this series we apply a potential of 1 volt.

We evaluate the voltage $v(k)$ at the point k in the circuit. By Ohm's law, if a resistor of r ohms is placed between two points x and y then the current, i_{xy}, flowing from x to y is $\frac{v(x)-v(y)}{r}$. Also by Kirchoff's law, the current flowing out of x equals the current flowing into x. Thus we have for our circuit in *Figure 1*, for every $k = 1, 2, \ldots, n-1$,

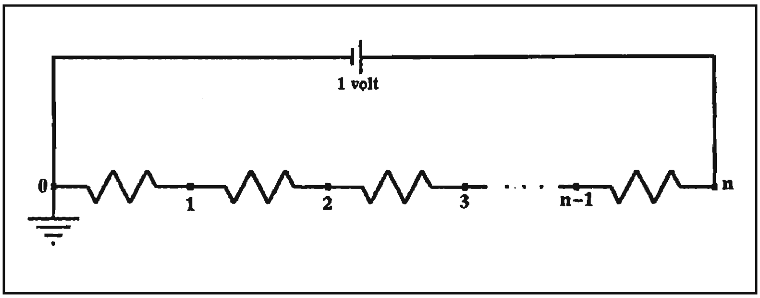


Figure 1.

$$\frac{v(k+1)-v(k)}{r} = \frac{v(k)-v(k-1)}{r},$$

which yields

$$v(k) = \frac{v(k+1)+v(k-1)}{2}. \tag{1}$$

Moreover, since the point 0 is grounded, we have at the endpoints 0 and n,

$$v(0) = 0 \text{ and } v(n) = 1. \tag{2}$$

These equations may be solved explicitly to yield

$$v(k) = k/n \text{ for } k = 0, 1, 2, \ldots, n. \tag{3}$$

Instead of connecting the resistors in a series, let us consider the circuit as in *Figure 2*.

Here the boundary is 'shorted', with the South-Eastern part of it grounded and shorted separately from the rest. We place resistors, each of r ohms, in the interior of the grid. If (k, l) denotes the coordinates of a point on the grid and $v(k, l)$ the voltage there, then application of Ohm's law and Kirchoff's law gives

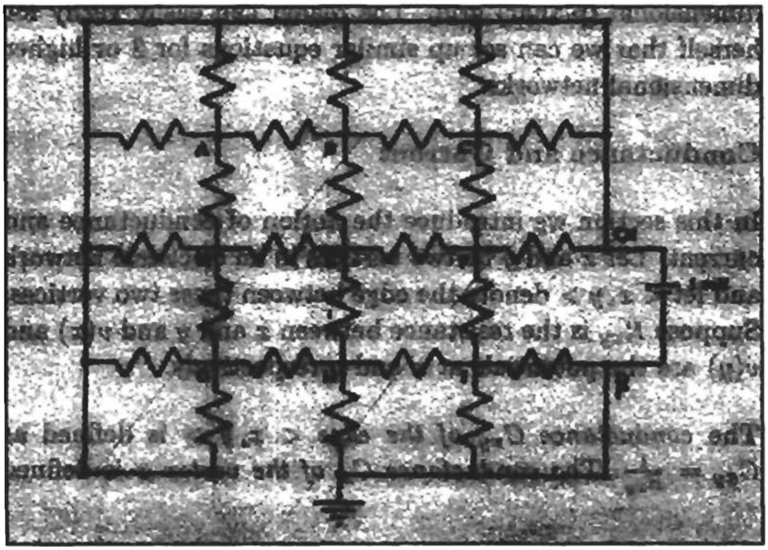


Figure 2.

$$\frac{v(k+1,l)-v(k,l)}{r}+\frac{v(k,l+1)-v(k,l)}{r}=$$

$$\frac{v(k,l)-v(k-1,l)}{r}+\frac{v(k,l)-v(k,l-1)}{r},$$

which yields

$$v(k,l)=\frac{v(k+1,l)+v(k-1,l)+v(k,l+1)+v(k,l-1)}{4}. \quad (4)$$

Also we have the boundary conditions

$$v(k,l)=\begin{cases}0 & \text{for } (k,l) \text{ on the South–Eastern boundary}\\ 1 & \text{for } (k,l) \text{ on the 'upper' boundary.}\end{cases} \quad (5)$$

These equations can be solved explicitly to yield

$$v=(v_A,v_B,v_C,v_D,v_E,v_F,v_G,v_H,v_I)=$$

$$(.9464,.9018,.9107,.8839,.7500,.7411,.8393,.4732,.3036),$$

where, $v_\alpha, v_\beta, v_A, v_B, \ldots, v_I$ represent the voltages at the points $\alpha, \beta, A, B, \ldots, I$ on the grid (as labelled in*Figure 2*).

We have till now dealt with resistors in series or on a 2-dimensional (planar) grid. The reader can easily verify for herself that we can set up similar equations for 3 or higher dimensional networks.

Conductance and Current

In this section we introduce the notion of conductance and current. Let x and y be two vertices in an electrical network and let $< x, y >$ denote the edge between these two vertices. Suppose R_{xy} is the resistance between x and y and $v(x)$ and $v(y)$ are the potentials at x and y respectively.

The *conductance* C_{xy} *of the edge* $< x, y >$ is defined as $C_{xy}=\frac{1}{R_{xy}}$. The *conductance* C_x *of the vertex* x is defined as $C_x = \sum_z C_{xz}$, where the sum is over all vertices z which are adjacent to the vertex x.

The *current* i_{xy} from x to y is defined as $i_{xy} = \frac{v(x)-v(y)}{R_{xy}}$, and thus,

$$i_{xy} = (v(x) - v(y))C_{xy}. \tag{6}$$

Now Kirchoff's law states that at any point x in the interior of the circuit (e.g. in *Figure 2* at any point other than α, β) $\sum_y i_{xy} = 0$, the sum being over all vertices y which are adjacent to the vertex x. Thus by (6), $\sum_y (v(x) - v(y))C_{xy} = 0$ and hence

$$v(x) = \sum_y v(y)\frac{C_{xy}}{C_x}. \tag{7}$$

The sums above are over all vertices y adjacent to the vertex x.

At the boundary points α and β of an electrical circuit, the current i_α flowing into the circuit is given by $i_\alpha = \sum_x i_{\alpha x}$, the sum being over all vertices x adjacent to the vertex α. Moreover, since the current flowing into the system equals the current flowing out of the system we have $i_\beta = -i_\alpha$ where $i_\beta = \sum_x i_{\beta x}$ and the sum is over all vertices x adjacent to the vertex β.

Now consider two circuits, one with boundary points α_1, β_1 and the other with boundary points α_2, β_2 such that $v(\alpha_1) = v(\alpha_2) = 1$ and $v(\beta_1) = v(\beta_2) = 0$ (i.e. β_1 and β_2 are grounded). Suppose the first circuit consists of a network of resistors between α_1 and β_1, while the second network consists of only one resistor of magnitude R ohms between α_2 and β_2. If $i_{\alpha_1} = i_{\alpha_2}$, i.e., the currents flowing into the two networks are the same, then from (6),

$$R = \frac{v(\alpha_2) - v(\beta_2)}{i_{\alpha_2}} = \frac{1}{i_{\alpha_2}} = \frac{1}{i_{\alpha_1}}. \tag{8}$$

Since the currents flowing into both the circuits are the same, the currents flowing out of the two circuits must also be the

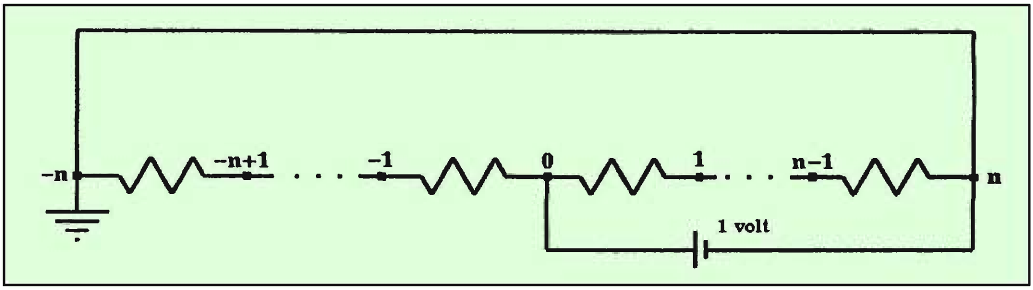


Figure 3.

same, thus the *effective resistance* ρ of the first circuit must equal that of the second circuit, i.e., $\rho = R$, and, by (8),

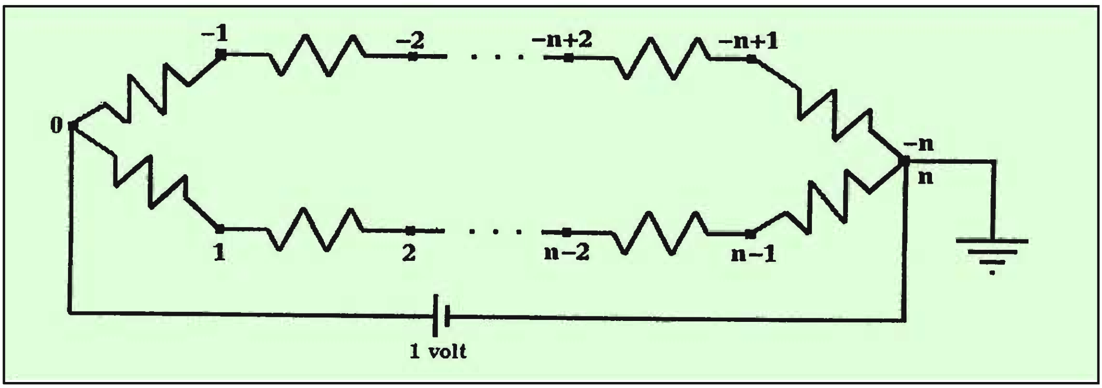
$$\rho = \frac{v(\alpha_1)}{i_{\alpha_1}} = \frac{1}{i_{\alpha_1}}. \tag{9}$$

Another way of calculating the effective resistance of a circuit is the one we learnt in high school, viz. two resistors of magnitudes r_1 and r_2 ohms
(a) connected in a series have an effective resistance of r_1+r_2 ohms,
(b) connected in parallel have an effective resistance of $1/(\frac{1}{r_1}+\frac{1}{r_2})$ ohms.

Towards Infinity

Suppose resistors, each of r ohms, are placed in a series as in *Figure 3*, with the vertices n and $-n$ shorted and grounded and a potential of 1 volt applied between 0 and the shorted 'boundary' n. This circuit is clearly equivalent to the circuit in *Figure 4*.

Figure 4.



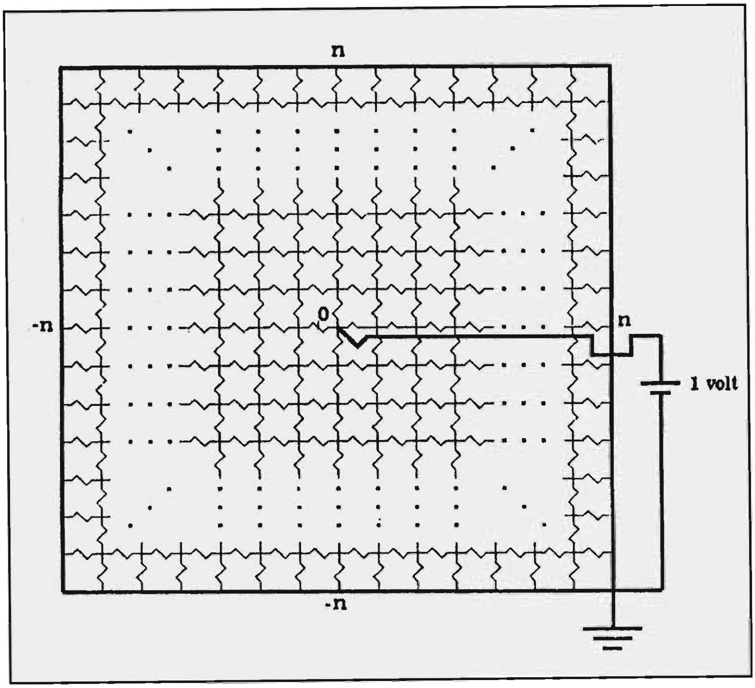


Figure 5.

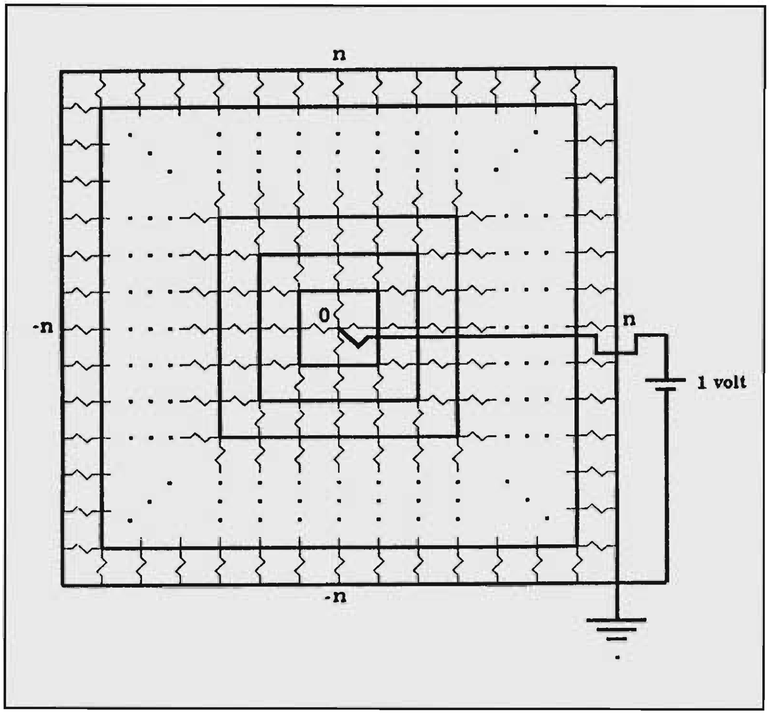
A simple calculation shows that the effective resistance ρ_n between 0 and n is $\frac{n}{2}r$. Clearly, $\rho_n \to \infty$ as $n \to \infty$. In other words, if a sequence of resistors is placed in a series with the series extending to infinity on both the left and the right sides, then the effective resistance between 0 and ∞ is infinite.

The equivalent problem for 2–dimensions is illustrated in *Figure 5.*

We place resistors each of r ohms on a grid of size $2n \times 2n$ centred at the origin 0 with the boundary of the grid shorted and grounded. We connect the origin 0 and the shorted boundary to a potential of 1 volt. Let ρ_n be the effective resistance of this circuit. As in the 1–dimensional case we will show that in this case too $\rho_n \to \infty$ as $n \to \infty$.

However, unlike in 1–dimension it is extremely difficult to obtain an exact expression for ρ_n. Indeed, we are also not interested in knowing the exact value of ρ_n, but only in show-

Figure 6.



ing that $\rho_n \to \infty$ as $n \to \infty$. To this end we begin by simplifying our work and consider the circuit as in *Figure 6*.

Here we short the boundary of each of the subsquares B_{k-1} of size $k \times k$, $k = 2, 3, \ldots, 2n$, centred at the origin 0 and connect resistors of r ohms each at the remaining edges of the grid. A potential of 1 volt is again applied between the origin 0 and the outermost shorted boundary.

But shorting an edge is equivalent to putting a resistor of 0 ohm between the vertices comprising the edge. In other words, the consequence of shorting is just to reduce the effective resistance of the circuit. Hence if ρ'_n is the effective resistance of the circuit in *Figure 6* and ρ_n is the effective resistance of the circuit in *Figure 5*, then

$$\rho'_n \leq \rho_n. \qquad (10)$$

The circuit in *Figure 6* is equivalent to the circuit in *Figure 7*,

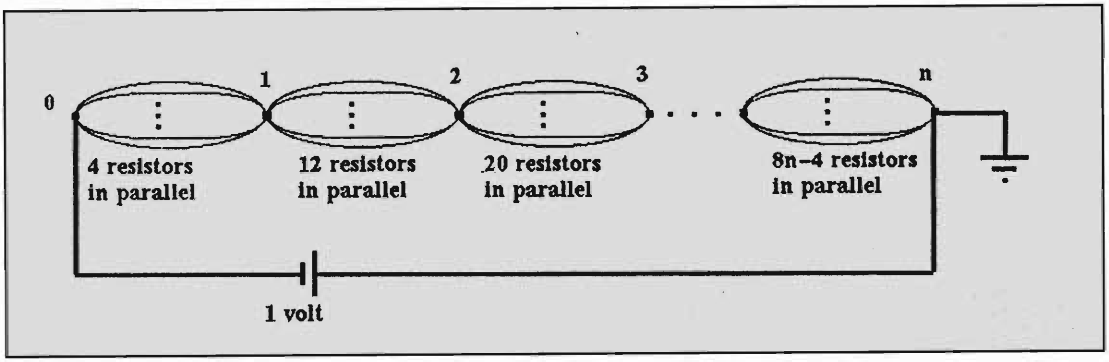


Figure 7.

where we have replaced each of the boundaries of the boxes B_k by a vertex k.

We observe that in the equivalent circuit (*Figure 7*) there are 4 resistors in parallel between 0 and 1, 12 resistors in parallel between 1 and 2, 20 resistors in parallel between 2 and 3, and in general, $8k - 4$ resistors in parallel between $k - 1$ and k. Clearly the effective resistance of the $8k - 4$ resistors in parallel between $k - 1$ and k is $r/(8k - 4)$ and so the effective resistance ρ'_n of the circuit in *Figure 6* satisfies

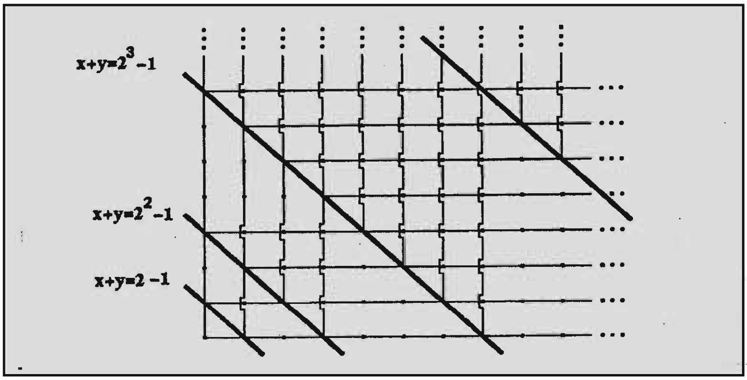
$$\rho'_n = \sum_{k=1}^{n} \frac{r}{8k-4} \to \infty \text{ as } n \to \infty.$$

Thus, from (10) we have for the circuit in *Figure 5*, $\rho_n \to \infty$ as $n \to \infty$.

The situation in 3 or higher dimensions is significantly different. Here we short the boundary of a cube of size $2n \times 2n \times 2n$ centred at the origin 0 and at each edge of the grid in this cube we place an r ohm resistor, while the shorted boundary is grounded. We apply a potential of 1 volt between the origin 0 and the shorted boundary. In this case we will show that the effective resistance ρ_n of the circuit does *not* go to infinity, instead we show that the effective resistance of the infinite grid in 3 or higher dimensions is finite.

Recall that in our argument for the 2–dimensional grid we first reduced the problem (*Figure 5*) to a simpler problem

Figure 8.



(*Figure 6*) by shorting the boundaries of $B_1, B_2, \ldots$, which ensured that the effective resistance ρ'_n of the circuit in *Figure 6* is less than the effective resistance ρ_n of the circuit in *Figure 5*. This method of shorting will not work in 3–dimensions. Indeed if we show the effective resistance of a simplified circuit obtained by shorting is finite, it will not be sufficient for our purposes because the effective resistance of this simplified circuit is less than that of the original circuit and hence it could still be the case that the effective resistance of the original circuit is infinite. Thus the finiteness of the effective resistance of a simplified shorted 3–dimensional grid does *not* imply the finiteness of the effective resistance of the 3–dimensional grid.

However all is not lost! Instead of shorting (i.e. connecting with a wire of zero resistance) we may remove an edge of the grid (i.e. connect with a wire of infinite resistance). If we were to do this then the 'simplified' circuit will have an effective resistance more than that of the original grid. Thus to establish the finiteness of the effective resistance of a three dimensional grid, we need to construct a suitable circuit from the original grid by removing edges and show that this simplified circuit has finite effective resistance.

The simplified circuit we employ is first described in 2–dimensions. (The reason for doing this is just that it is easier to draw 2-dimensional pictures!) We start from the origin 0 and draw two lines – one going up and the other going to the right (see *Figure 8*). As soon as either of these lines meet the line $x + y = 2^n - 1$, $n = 1, 2, \ldots$, it bifurcates into two lines – one going up and the other going right, which in turn bifurcate when they meet the line $x + y = 2^n - 1$. We continue this method of connecting edges between the vertices of the 2-dimensional grid. In the electrical network, each edge corresponds to a r ohm resistor and so each of these lines consists of a string of r ohm resistors in series, and we ensure that except at the point of bifurcation two distinct strings of resistors are not connected.

It is immediately clear that the effective resistance of this circuit is more than that of a two dimensional grid because not connecting two points by a resistor is equivalent to putting a resistor of infinite magnitude between the two points. This clearly increases the effective resistance of the resulting circuit. Thus the effective resistance of the circuit in *Figure 8* is more than that of the 2-dimensional infinite grid.

By redrawing the circuit in *Figure 8* and following each string of resistors we see that *Figure 8* is equivalent to the tree in *Figure 9.*

It is easy to calculate the effective resistance of this tree by observing that the symmetry of the tree ensures that each of the 2^k points in generation k have the same potential and so we can connect each of these 2^k points without changing the effective resistance of the circuit. Hence the tree in *Figure 9* is equivalent to the circuit in *Figure 10.*

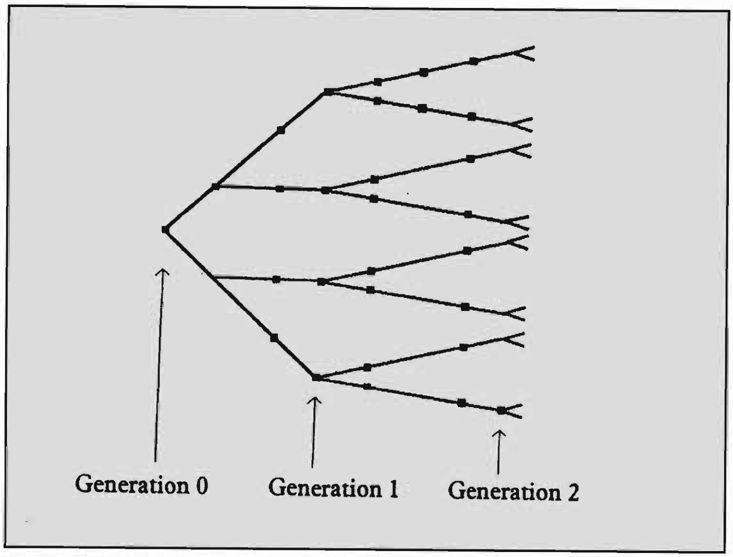


Figure 9.

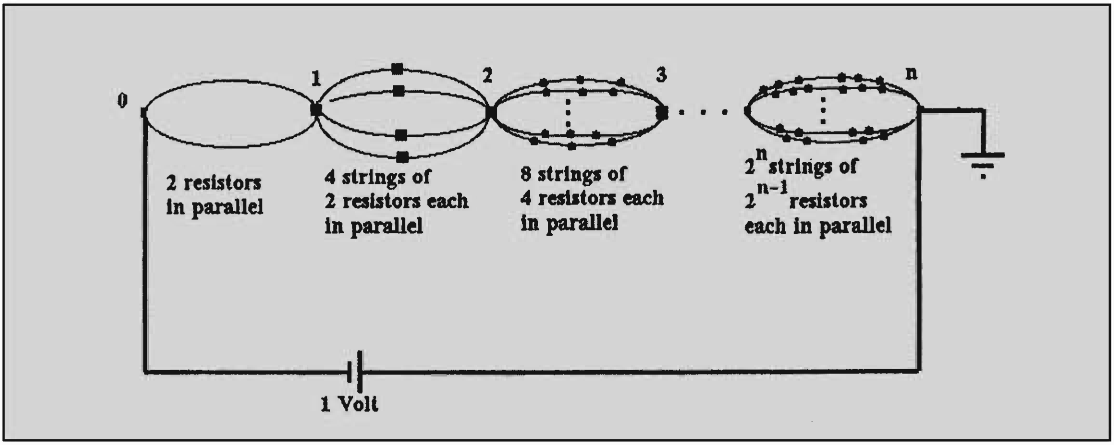


Figure 10.

A simple calculation now shows that the effective resistance ρ'_n of this circuit upto generation n is $\frac{n}{2}r$ which goes to infinity as $n \to \infty$. This is, of course, not surprising because we had already noted earlier in *Figure 7* that the effective resistance of the 2–dimensional grid is infinite.

We will do a similar construction for 3 dimensions. We will not attempt to draw a figure here because this page lacks a dimension necessary to do justice to a figure and so we describe the procedure in words. We start from the origin and draw 3 lines going along the x, y and z axes respectively. When any of these lines meet the plane $x+y+z = 2^n-1$ (for some $n = 1, 2, 3, \ldots$) it bifurcates into three lines – one going parallel to the x axis, another going parallel to the y axis and the third going parallel to the z axis – each of which in turn bifurcate when they meet the plane $x + y + z = 2^n - 1$ (for some $n = 1, 2, 3, \ldots$). We continue this method of connecting edges between the vertices of the 3-dimensional grid. In the electrical network, each edge corresponds to a r ohm resistor and so each of these lines consists of a string of r ohm resistors in series, and we ensure that except at the point of bifurcation two distinct strings of resistors are not connected.

Again, redrawing the circuit obtained by the above procedure and following each string of resistors we see that this circuit is equivalent to the tree in *Figure 11*.

The effective resistance can be calculated for this tree by observing that each of the 3^k vertices at generation k have the same potential and hence we can connect each of these 3^k vertices. The effective resistance ρ'_n of this circuit upto generation n is

$$\frac{r}{3}+\frac{r}{\frac{9}{2}}+\frac{r}{\frac{27}{4}}+\cdots+\frac{r}{\frac{3^n}{2^{n-1}}}=[1-(\frac{2}{3})^n]r, \qquad (11)$$

which goes to r as $n \to \infty$.

Since the 3–dimensional grid has an effective resistance less than or equal to that of the above simplified circuit, it has an effective resistance of at most r.

For dimensions $d > 3$, note that by removing all resistors from $d-3$ dimensions we obtain a 3–dimensional grid whose effective resistance we know to be finite. Since removing all resistors from $d-3$ dimensions is equivalent to substituting each of these resistors by ∞ ohm resistors, we see that the effective resistance of a d–dimensional grid is always less than or equal to that of a 3–dimensional grid. Hence the effective resistance of a d–dimensional grid is finite for all $d \geq 3$.

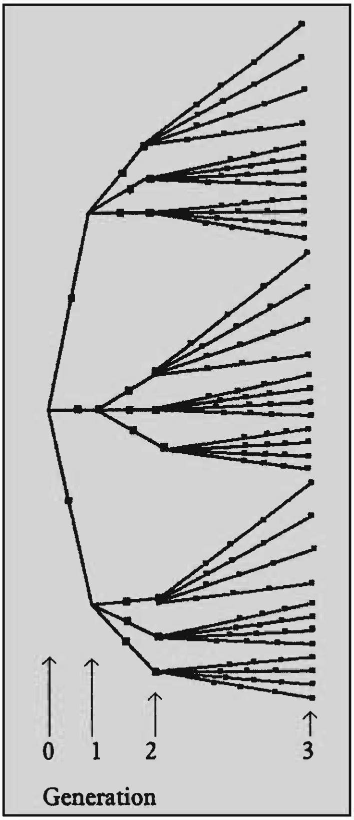


Figure 11.

Suggested Reading

- Polya G. *Uber eine aufgabe der wahrscheinlichkeitsrechnung bettreffend die irrfahrt in strassennetz. Mathematische Annalen.* Vol.84. pp. 149–160, 1921.
- Doyle P and Snell J L. Random walk and electrical networks. *Carus mathematical monograph* No. 22. American Mathematical Association.Washington. DC, 1984.

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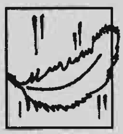

The goal of science is to build better mousetraps.
The goal of nature is to build better mice.

Anonymous