Think It Over

This section of Resonance is meant to raise thought-provoking, interesting, or just plain brain-teasing questions every month, and discuss answers a few months later. Readers are welcome to send in suggestions for such questions, solutions to questions already posed, comments on the solutions discussed in the journal, etc. to Resonance Indian Academy of Sciences, Bangalore 560 080, with “Think It Over” written on the cover or card to help us sort the correspondence. Due to limitations of space, it may not be possible to use all the material received. However, the coordinators of this section (currently R Nityananda and C S Yogananda) will try and select items which best illustrate various ideas and concepts, for inclusion in this section.

1. Buffon’s Needle Problem

Consider a plane ruled with parallel lines that are a distance \( D \) apart. Take \( m \) needles of length \( L \) each, \( L \leq D \). Drop them all randomly on this plane. Count the number of needles that come to rest crossing any one of the lines. Repeat this \( n \) times. Let \( x_1, x_2, \ldots, x_n \) be the number of needles that cross lines in these \( n \) trials. If you are told that, for large values of \( m \) and \( n \), \( \frac{2L}{D} \sum_{i=1}^{mn} x_i \) will be close to \( \pi \) what will be your reaction?

Correct answers have been received from Bikramjit Banerjee and Soubhik Chakraborty. A sketch of the solution follows.

Consider just two parallel lines and just one needle as shown in the figure. Let \( X \) denote the distance of the point of rest of the needle from the bottom line. Then \( X \) has the uniform distribution on the interval \((0, D)\). Let \( \theta \) denote the angle of rest. Then \( \theta \) has the uniform distribution on \((0, 2\pi)\). The probability that the needle comes to rest crossing one of these two lines is precisely \( P(L \sin(\theta) > X, 0 < \theta \leq \pi/2) + P(L \sin(\pi - \theta) > X, \pi/2 < \theta \leq \pi) + P(L \sin(\theta - \pi) > D - X, \pi < \theta \leq 3\pi/2) + P(L \sin(2\pi - \theta) > D - X, 3\pi/2 < \theta \leq 2\pi) \). Now,
\[ P(L \sin(\theta) > X, 0 < \theta \leq \pi/2) \]

\[ = \frac{1}{2\pi D} \int_{0}^{\pi/2} \int_{0}^{D} I_{\{L \sin(\theta) > x\}} \, d\theta \, dx \]

\[ = \frac{1}{2\pi D} \int_{0}^{\pi/2} L \sin(\theta) \, d\theta = \frac{L}{2\pi D}, \]

and, in fact, the other three probabilities are also equal to the same quantity, \( \frac{L}{2\pi D} \). Therefore, the probability that any needle dropped randomly on this plane comes to rest crossing any of the lines is \( \frac{2L}{\pi D} \).

Next, recall the Law of Large Numbers (see the article by R L Karandikar, ‘On Randomness and Probability’, Resonance, Vol.1. No.2. pp.55-68, February 1996). Note that, if \( x_i \) denotes the number of needles (out of \( m \) dropped) that come to rest crossing lines in the \( i \)th trial, then \( \sum_{i=1}^{n} x_i \) is the total number of needles that come to rest crossing lines out of \( mn \) needles which are dropped randomly and independently on the plane. Therefore, for large values of \( mn \), \( \sum_{i=1}^{n} x_i/mn \) will be close to \( \frac{2L}{\pi D} \), which is the probability that any needle dropped randomly on this plane comes to rest crossing any of the lines.