



In this section of Resonance, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

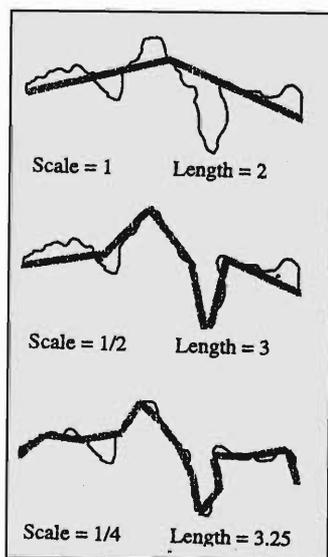
Balkrishnan Ramasamy,
T S K V Iyer, P Varadharajan
BITS, Pilani.

Fractals: A New Geometry of Nature

What is the Length of a Coastline?

This seemingly innocuous question was posed by Louis Richardson as "how long is the coastline of Britain?" in 1961. Richardson was studying the border between Spain and Portugal, which had been the source of dispute between the two nations. He found that in Spanish encyclopaedias the border was 987 km long whereas the Portuguese encyclopaedia put the figure at 1214 km. Richardson argued that the discrepancy is not due to inaccuracy or error, but the question itself needs more investigation.

Figure 1. Length of a coastline measured by decreasing scale lengths.



We will now consider the problem of finding the length of a piece of coastline shown in *Figure 1*. Take a scale of length 1 unit and measure the coastline by walking along it. The length of the coast turns out to be 2 scale-lengths and since the length of the scale we used was of 1 unit, the coastline is 2 units long. We will now measure the length of the same coastline, but will use a scale of length 1/2 unit. The coast is covered by 6 scale-lengths of 1/2 unit each, so the length now turns out to be 3 units. The length has increased because the projections and depressions that were missed out in the first measurement come into the picture when we reduce the

length of the scale resulting in an increase in the length of the coastline. When we reduce the scale-length further to $1/4$ unit, the length of the coastline becomes 3.25 units. Richardson argued that the length of a coastline will increase indefinitely as we decrease the length of the scale used to measure it because bays and peninsulas will reveal smaller sub-bays and sub-peninsulas and so on. In other words, the length of a coastline depends very much on the length of the scale used to measure it!

Suppose we measure the length of a straight line of 2 units length (when measured by a scale of length 1 unit) by decreasing the scale lengths. If we reduce the length of the scale to $1/2$ unit, the length will still be 2 units because we can cover the line with 4 scale-lengths. In contrast to a coastline, decreasing the length of the scale has no effect on the length of a straight line. For circles and smooth arcs, the length will converge to a value instead of increasing indefinitely.

To understand the concept of indefinitely long coastlines, we will construct an artificial coastline (see *Figure 2*) according to the following procedure:

- Take a straight line and divide it into three equal parts.
- Remove the middle one-third of the line and replace it

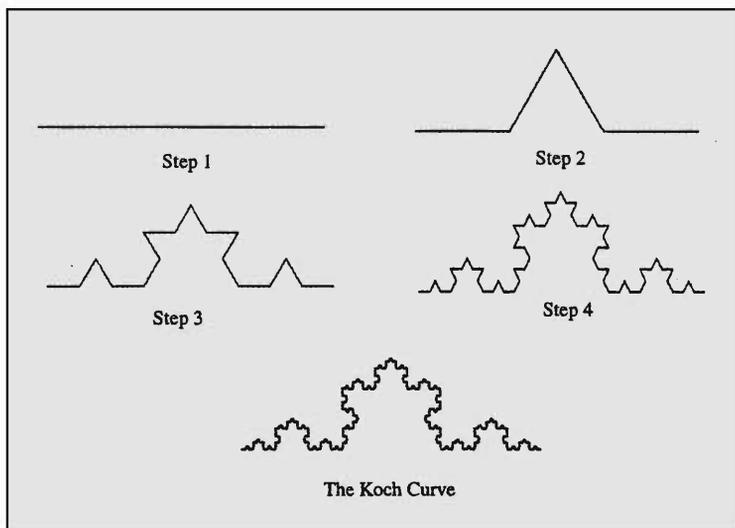


Figure 2. The Von Koch curve is a rough model of a natural coastline.

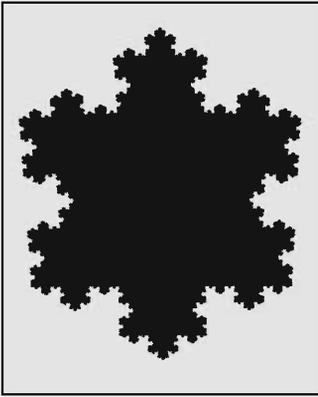


Figure 3. The Von Koch snowflake. The initiator is an equilateral triangle.

with two line segments as shown in step 2.

- Perform the above step on each resulting line segment.

We shall now measure the length of the artificial coastline generated by the above procedure. If we take a scale of length 1 (which is exactly as long as the line segment in step 1 of Figure 2), we find that the length of the coastline is also one. If we reduce the scale length to $1/3$, the length of the coastline becomes $4/3$, because in step 2 of the generation process, we have replaced the middle one-third of the original line with two line segments. As we keep decreasing the scale length, the length of the artificial coastline also increases indefinitely. This is easily seen because at each step of the generation process we have increased the length of the coastline by four-thirds.

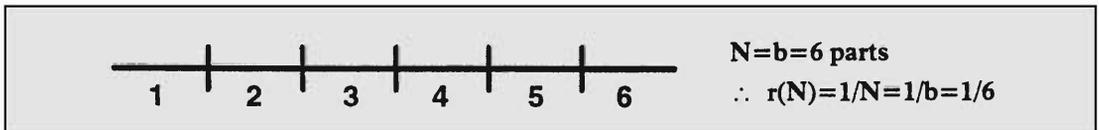
The artificial coastline described above, called the *Koch curve*, is a *fractal*. If we magnify any part of the Koch curve, we will observe that it is similar to the parent curve. This property of *self-similarity* is shared by all the fractals.

In fractals parlance, the initial line segment of the Koch curve is called the *initiator* and the operation that we perform on each line segment, represented by step 2 of the generation procedure is called the *generator*. A close relative of the Koch curve is the Koch snowflake (Figure 3) (or is it the Koch island?) generated using the same procedure described above except that we use an equilateral triangle, instead of a straight line as the initiator.

Fractal Dimension

Figure 4. Similarity ratio $r(N)$ of a straight line.

Consider a straight line. Its dimension is 1. Suppose we want to cover the whole line by $N = b$ smaller parts (Figure 4). It



is evident that each smaller part can be deduced from the whole by a similarity ratio $r(N) = 1/b = 1/N$. Roughly, we can think of the similarity ratio $r(N)$ as the ratio of the length of the smaller line segment to the length of the original line segment.

Now, consider a square. Its Euclidean dimension is 2. Let us cover it with $N = b^2$ squares, where b is the length of the side of the smaller squares. The similarity ratio $r(N) = 1/b = 1/N^{1/2}$ is the ratio of the length of a side of the smaller squares to the length of a side of the bigger square, which the smaller ones cover. This is illustrated in Figure 5. Note that the similarity ratio can also be applied to other two-dimensional shapes like rectangles and triangles.

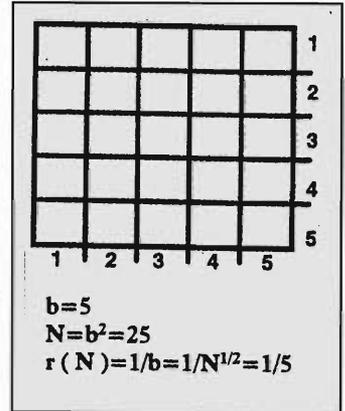


Figure 5. Similarity ratio of a square.

In general for a D-dimensional Euclidean object the similarity ratio is

$$r(N) = 1/N^{1/D} \tag{1}$$

From the above expression, a formula for the dimension D in terms of the similarity ratio r and the number of parts N can be derived as

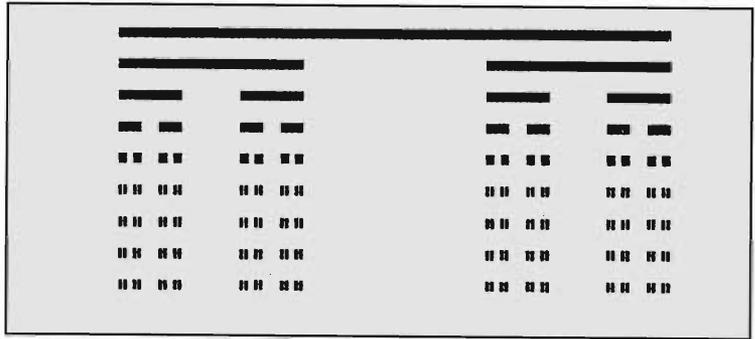
$$D = \log N / \log (1/r) \tag{2}$$

Definition : If an object can be decomposed into N sub-objects each of which is exactly like the whole thing except that all the sub-objects are derived from the original by a similarity ratio r , then the whole object is exactly self-similar, and the similarity dimension D of the object is given by (2).

It is evident that the Koch curve is self-similar. So for the Koch curve, for a similarity ratio of $1/3$, we have 4 parts. Hence $D = \log 4 / \log 3 = 1.2618$ (approx.). Note that the dimension of the Koch curve lies between that of a line (dimension 1) and a plane (dimension 2).



Figure 6. The Cantor set, one of the simplest of fractals.



Some Fractals

Cantor Set: The Cantor set (*Figure 6*) is a very simple fractal constructed as follows. Take a line segment and divide it into three parts. Remove the middle one-third of the line segment. Perform the previous operation on each of the remaining two line segments. Continue the process ad infinitum. In *Figure 6*, after a few steps, the line segments have become so small that subsequent divisions are invisible. The end result is an uncountable, unconnected and infinite number of points in place of the line segment. The Cantor set is named in honour of Georg Cantor, who devised the set to demonstrate that real numbers are uncountably infinite.

Since at each step of the iteration we are dividing the line segment into 3 parts ($r = 1/3$) and replacing it with two line segments ($N = 2$), the similarity dimension or the fractal dimension of the Cantor set is $D = \log 2 / \log 3 = 0.6309$.

Sierpinski Triangle

The Sierpinski triangle is constructed by taking an equilateral triangle as an initiator. The generation process is achieved by removing the triangle formed by joining the midpoints of the sides of the equilateral triangle. The generator is then applied to each of the three newly created triangles. The steps in the generation of the Sierpinski triangle are illustrated in *Figure 7*. The dimension of the Sierpinski triangle is calculated as



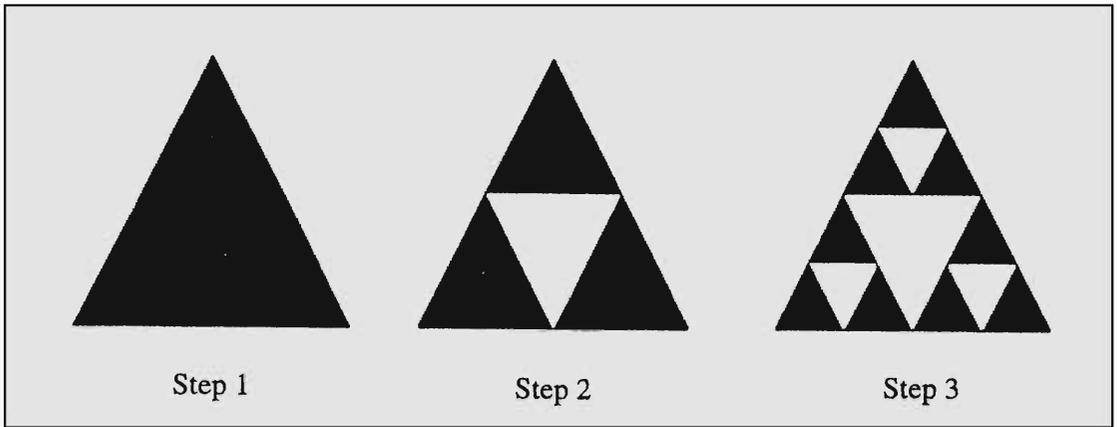
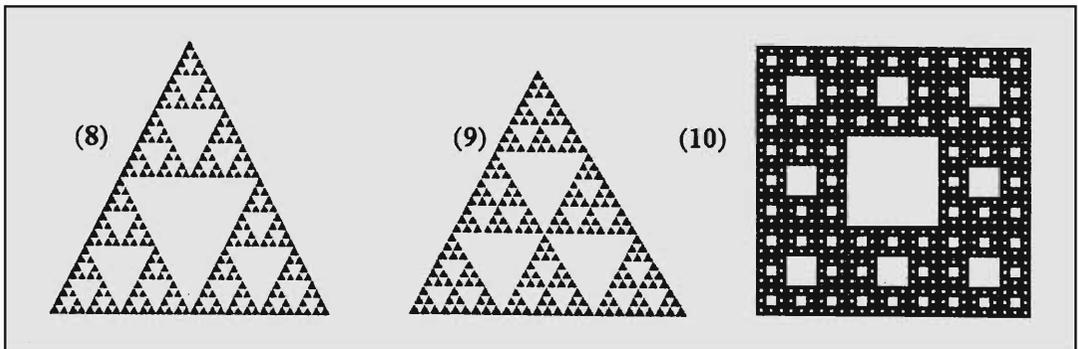


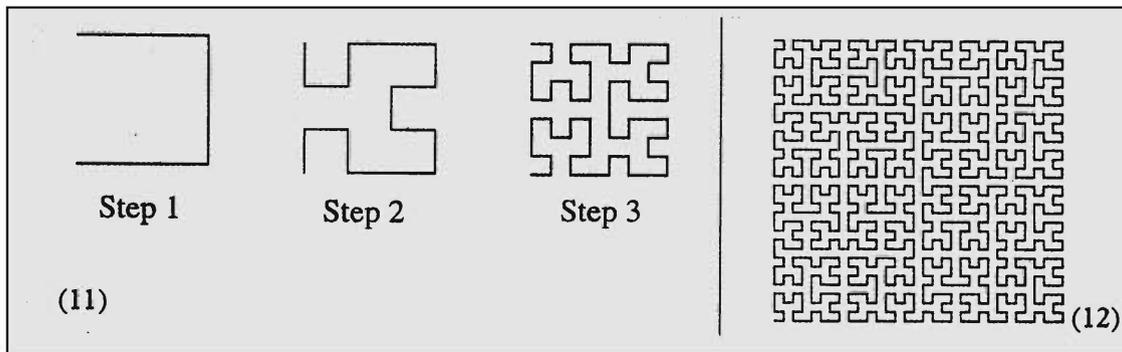
Figure 7. Steps in the generation of the Sierpinski triangle.

follows. The length of the side of the smaller triangle is $1/2$ that of the triangle of the previous iteration. The triangle of the previous iteration can be paved by 4 such smaller triangles of which the middle one is removed. Therefore $N = 3$. Applying (2), the dimension of the Sierpinski triangle is $D = \log 3 / \log 2$. The Sierpinski triangle obtained after a few iterations is shown in *Figure 8*.

Figure 9 shows a variation of the Sierpinski triangle, in which three smaller triangles are removed from the larger triangle. Can you find out the dimension of the Sierpinski triangle variant? The Sierpinski carpet (*Figure 10*) is generated by taking a square, dividing it into nine squares and removing the central smaller square, and repeating the process for each of the remaining squares. What is the fractal dimension of the Sierpinski carpet?

Figures 8-10. (left) The Sierpinski triangle. (center) Variant of the Sierpinski triangle. (right) The Sierpinski carpet.





Figures 11 and 12. (left) Steps in the generation of the Hilbert curve. (right) Hilbert curve after a few steps.

Are Fractals 'Pathological' ?

Though fractal geometry is a fairly recent science, most of the fractal shapes have been around for a century at least. The Cantor set, Koch curve, Sierpinski triangle etc. are named after the turn-of-the-century mathematicians who constructed them.

Suggested Reading

- ◆ B Mandelbrot. *The Fractal Geometry of Nature*. W H Freeman. New York, 1982.
- ◆ H O Peitgen and P Richter. *The Beauty of Fractals*. Springer-Verlag. New York, 1986.
- ◆ James Gleick. *Chaos : Making a new Science*. Penguin. New York, 1987.
- ◆ H Jurgens, H O Peitgen and D Saupe. The Language of Fractals. *Scientific American*. 263. Vol. No. 2. pp. 40-47, Aug. 1990.
- ◆ H O Peitgen, H Juergens and D Saupe. *Fractals for the Classroom (Vol. I & II)*. Springer-Verlag. New York, 1992.

Fractal curves do display some strange properties at first sight. Take the Koch snowflake (*Figure 3*) as an example. The curve is infinitely long (because we keep increasing the length of the snow flake by $4/3$ in each step of the generation process) and yet is confined within a finite area and never exceeds a circle drawn around it. The curve never intersects itself (self-avoiding) and is also not differentiable at any point. These properties perplexed the mathematicians of the time. They called them 'pathological' curves.

One such 'pathological' structure is the Hilbert curve (*Figure 11*). The curve has a fractal dimension of 2, which means that if we carry on the generation process to infinity it will cover up the whole plane. Such curves are called 'space-filling curves'. The Hilbert curve after a few steps is shown in *Figure 12*. Though the dimension of the Hilbert curve is 2, it is also a fractal.

