The Isoperimetric Problem [1]

"Among all simple closed curves[2] γ on the plane having a fixed perimeter, the circle encloses the maximum area". This is the celebrated isoperimetric problem and one of the proofs (due to Hurwitz) uses results about Fourier series for this purely geometrical statement! [An equivalent statement is: "Among all simple closed curves γ on the plane enclosing a given area, the circle has the least perimeter". (Exercise: Why are the two statements equivalent?)] For the necessary background on Fourier series, the reader can consult the article by S Thangavelu in *Resonance*, October 1996.

[2] By a simple closed curve on the plane, we mean the path traced by a point p on the plane, which is initially at p_0, at time $t = a$, and returns to p_0 at a later time $t=b$ *without* crossing its path at any other time between a and b. To avoid pathologies, we assume that all our curves are *rectifiable* i.e. have *finite length*! This is called the *perimeter* of the curve.

In fact, the statement above is a consequence of the following inequality:

$$L^2 - 4\pi A \geq 0, \tag{1}$$

where L is the *perimeter* of the simple closed curve and A is the area enclosed, and *equality is attained in the above if and only if γ is a circle.* (Exercise: Convince yourself that this implies the assertion made in the beginning of the first paragraph.)

We make certain simplifying assumptions that are not really very serious. Let us assume the curve γ is traced out by a moving point p whose coordinates are $(\gamma_1(t), \gamma_2(t))$. Let us also assume that at time $t = 0$, we are at position p_0 and that we are back at the same position at time $t = 2\pi$ (see *Figure 1*).

By repeating this over and over again, we can think of $\gamma_1(t)$ and $\gamma_2(t)$ as periodic functions of t with period 2π. To simplify matters, we take $\gamma_1(t)$ and $\gamma_2(t)$ to be *smooth* functions, in particular $\gamma_1'(t)$ and $\gamma_2'(t)$ exist and are continuous functions. (The case of *non-smooth* curves can be taken care of by an appropriate approximation procedure.) Since the perimeter L and the area A enclosed do not depend on how exactly we move around the curve, let us move in

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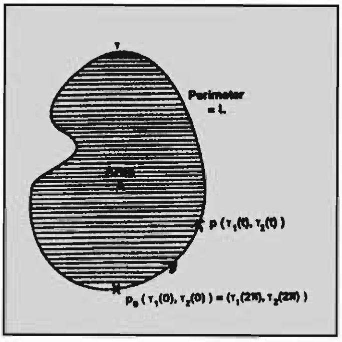

Figure 1

an *anti-clockwise manner with constant speed u.* Thus $(\gamma_1'(t))^2 + (\gamma_2'(t))^2 = u^2$, for all t. But, since the total length is L and each cycle is completed in 2π units of time, $u = \frac{L}{2\pi}$ and so we have $\frac{L^2}{4\pi^2} = (\gamma_1'(t))^2 + (\gamma_2'(t))^2$ for all t.

Thus,

$$\int_0^{2\pi} \frac{L^2}{4\pi^2}\,dt = \frac{L^2}{2\pi} = \int_0^{2\pi} [(\gamma_1'(t))^2 + (\gamma_2'(t))^2]\,dt. \tag{2}$$

Since, as has already been pointed out, $\gamma_1(t)$ and $\gamma_2(t)$ are 2π-periodic functions of t, we can express them in the form of *Fourier series:*

$$\gamma_1(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt, \tag{3A}$$

$$\gamma_2(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos nt + \sum_{n=1}^{\infty} B_n \sin nt. \tag{3B}$$

So :

$$\gamma_1'(t) = \sum_{n=1}^{\infty} nb_n \cos nt + \sum_{n=1}^{\infty} (-na_n) \sin nt, \tag{3C}$$

$$\gamma_2'(t) = \sum_{n=1}^{\infty} nB_n \cos nt + \sum_{n=1}^{\infty} (-nA_n) \sin nt. \tag{3D}$$

Hence, using basic facts about Fourier series,

$$\int_0^{2\pi} (\gamma_1'(t))^2\,dt = \pi \sum_{n=1}^{\infty} (n^2 b_n^2 + n^2 a_n^2)$$

and

$$\int_0^{2\pi} (\gamma_2'(t))^2\,dt = \pi \sum_{n=1}^{\infty} (n^2 B_n^2 + n^2 A_n^2).$$

Thus from the above and (2) we get

$$\frac{L^2}{2\pi} = \pi \sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2 + A_n^2 + B_n^2). \tag{4}$$

Joseph Fourier (1768-1830)

On the other hand, from our knowledge of *advanced* calculus, we know that the enclosed area is given by the formula:

$$A = \int_0^{2\pi} \gamma_1(t)\gamma_2'(t)\,dt.$$

(The reader is encouraged to look up a proof of this result which relies on Green's theorem.) But again from our knowledge of Fourier series and using (3A) and (3D) we have

$$\int_0^{2\pi} \gamma_1(t)\gamma_2'(t)\,dt = \pi \sum_{n=1}^{\infty} n(a_nB_n - A_nb_n).$$

So,

$$A = \pi \sum_{n=1}^{\infty} n(a_nB_n - A_nb_n). \tag{5}$$

Using (4) and (5) (and some simple school algebra!), we get

$$L^2 - 4\pi A = 2\pi^2 \sum_{n=1}^{\infty} \Big[(na_n - B_n)^2 + (n^2-1)B_n^2 + (nb_n + A_n)^2 + (n^2-1)A_n^2\Big] \tag{6}$$

The quantity on the right in (6) is clearly non-negative! Hence $L^2 - 4\pi A \geq 0$ and the proof of inequality (1) is complete.

When is equality attained in (1)? For this, the sum of the infinite series in (6) must be zero. Since each term of the infinite series is non-negative and each of the four quantities in each term is non-negative, we are led to:

$$na_n = B_n, \quad (n^2-1)B_n^2 = 0,$$
$$nb_n = -A_n, \quad (n^2-1)A_n^2 = 0,$$
$$\text{for } n = 1, 2, \cdots.$$

Thus $B_n = 0, \quad A_n = 0$ for $n \geq 2$.

Also, $a_1 = B_1$ and $b_1 = -A_1$ and if we let $a_1 = B_1 = C$ and $b_1 = -A_1 = D$, from (3A) and (3B), we have

$$\gamma_1(t) = a_0 + C\cos t + D\sin t$$

$$\gamma_2(t) = A_0 - D\cos t + C\sin t$$

and so $(\gamma_1(t) - a_0)^2 + (\gamma_2(t) - A_0)^2 = C^2 + D^2$. This is clearly the equation of the circle with centre at (a_0, A_0) and radius $\sqrt{C^2 + D^2}$, and the proof of the assertion we started with is complete!

Though much simpler proofs of inequality (1) are available (see Suggested Reading, for instance the article of P D Lax), the method described in this note actually provides more information about L and A in terms of the Fourier coefficients (via equation (6)) for an *arbitrary closed curve* γ. This kind of information, linking purely *geometric data* (like L and A) with purely *analytic data* (like the Fourier coefficients), is considered very important in mathematics. For some exciting connections between *analysis* and *geometry*, we refer the reader to the beautiful (but difficult!) article by M Kac (see Suggested Reading). We conclude by pointing out that the following three dimensional analogue of the question we started with is also true (although more difficult to prove): Among all closed surfaces enclosing a given volume, the sphere has the smallest surface area. (Ask your physics teacher about the important implications of this proposition to physics!)

Suggested Reading

- M Kac. Can one hear the shape of a drum? *American Mathematical Monthly*. Vol.73. pp.1-23, 1966.
- T W Körner. *Fourier series*. Cambridge University Press, 1989.
- P D Lax. A short path to the shortest path. *American Mathematical Monthly*. Vol.102. pp.158-159, 1995.
- S Thangavelu. Fourier series. *Resonance*. Vol.1. No.10. pp.44-55, October 1996.

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