Projective Geometry

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The following is a write-up of a talk that was presented at the TIFR as part of the Golden Jubilee celebrations of that Institute during 1996.

Introduction

For some reason not so well understood, mathematicians find it most difficult to communicate their joys and frustrations, their insights and experiences to the general public, indeed even to other fellow scientists! Perhaps this is due to the fact that mathematicians are trained to use very precise language, and so find it hard to simplify and state something not entirely correct. You must have heard of the gentleman who while he was drowning, exclaimed, "Oh! Lord, if You exist! Help me, if You can!" I am sure it was a mathematician. On the other hand, mathematics carries a certain mystique for most non-mathematicians. It is one subject to which nobody seems to have been neutral at school. Everyone says, "I loved mathematics at school", or "I hated it". Nobody says it was just one of those subjects. Let me leave it to psychologists to analyse why it is so and get on with my subject matter.

With some concessions on your part for some resultant vagueness, I will try to communicate some ideas basic to the topic which I have been interested in for a considerable percentage of my professional life, namely, Projective Geometry.

Perspective

Mathematical theories are, by and large, off-shoots of applications and not precursors. Whatever the validity of such a general formulation, it is certainly true of Projective Geometry. The first seeds of this theory may be seen in the attempt to understand *perspective* in painting. The non-triviality of this feeling for perspective is clear when one sees the paintings of the Persians, Hindus or the Chinese (see *Figure 1*). These are stylistic no doubt, but the lack of perspective is not deliberate, as in Cezanne (for example). Contrast this with the paintings made during the renaissance in Europe (see *Figure 2*). The perspective is obvious in these. In the figure on the right the lines have been extended (produced) to show you that all of them meet at a point.

Figure 1 Painting without perspective.

Let us try to find out what the feeling of depth that this painting creates is due to, from the mathematical point of view. Look at the floor in the painting. You see the lines formed by the tiles there? You all know that these are parallel lines. But they do meet, and what is more, if you extend these lines, all these mutually parallel lines meet at the same point. In painting, this point is sometimes referred to as a 'vanishing point'. We will get back to this device called 'central projection' in mathematics a little later.

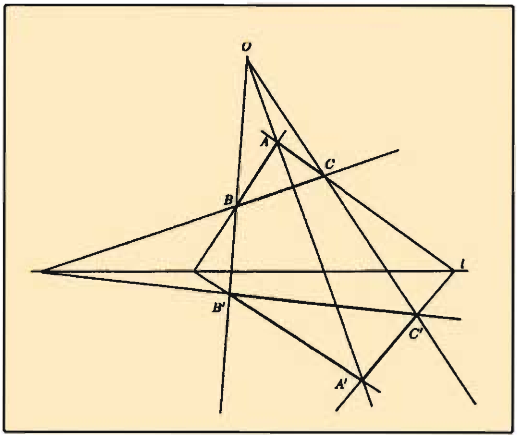
So you will not be surprised when I tell you that the first theorem on projective geometry is due, not to a mathematician, but a professional engineer called Girard Desargues. It is a sound principle that in any mathematical lecture there should be at least one theorem stated and proved! So, let me state

Desargues' Theorem

The assumptions are that A, B, C and A', B', C' are two triples of points such that the lines AA', BB' and CC' all pass through a point O (see *Figure 3*). Then one says that the two triangles are in *perspective*. Let us examine another figure with two triangles in perspective,

Figure 2 Paintings with perspective.

Figure 3 Desargues' Theorem.



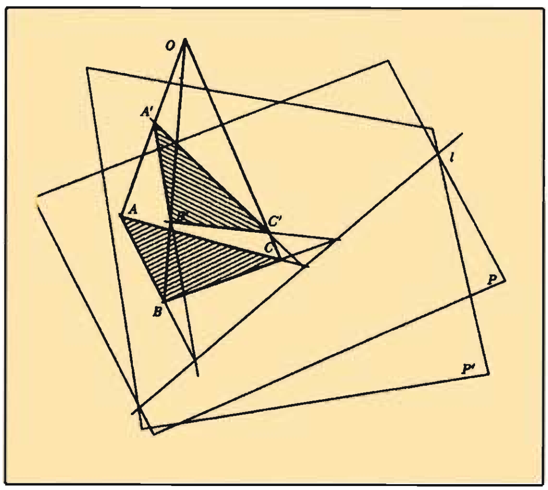
but now in three dimensions (see *Figure 4*). Here you have a tetrahedron which has been sliced by two planes giving us two triangles. Note first that the lines BC and $B'C'$ meet. Although two lines in three-space need not intersect, these two lie on one face of the tetrahedron and so do meet. Similarly the other two sides. The conclusion is that the point where the lines BC and $B'C'$ meet, the point where CA and $C'A'$ meet and the point where AB and $A'B'$ meet, all lie on a line.

Actually the theorem, surprising as it may seem, is trivial if one uses three dimensions rather than two. Let l be the line of intersection of the two slicing planes. Then BC and $B'C'$ lie in these planes and so their intersection lies in the intersection of the two slicing planes. The same holds for the other three points. So we have proved the theorem.

To prove it in two dimensions, one may lift one of these triangles suitably in space, and use it to prove the theorem. I shall content myself with showing you an illustration (see *Figure 5*) of how this is done. Indeed, one can show that within the framework of projective geometry, the theorem cannot be proved without the use of the third dimension!

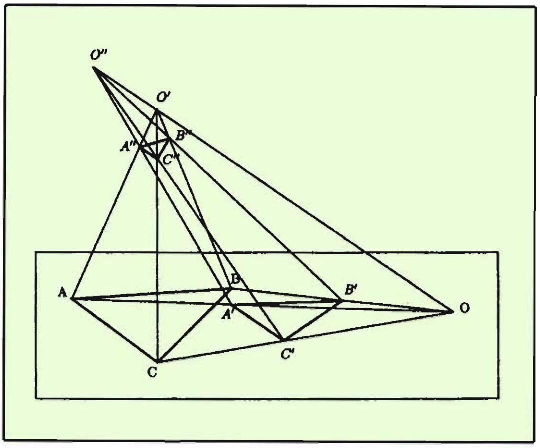
As a rule, the Euclidean theorems which most of you have seen would involve angles or lengths. This theorem does not use any such thing. If you take a plane on which there are two such triangles which enjoy the above property and centrally project from a point then you get again two such triangles. Such properties are called *projective* properties.

Figure 4 Desargues' Theorem in 3 dimensions.



Actually Desargues proved the theorem because he was interested in perspective drawings and the geometry it gave rise to is however *not* projective geometry but what is called *descriptive geometry*. I understand this is

Figure 5 Lifting of one of the triangles.



useful to engineers even today.

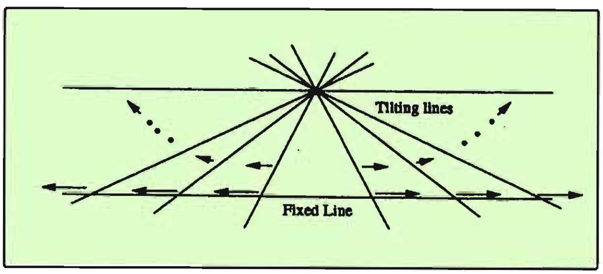
As far as projective geometry itself is concerned, although some ideas may be traced to Monge, Pascal, Chasles, Plücker, Steiner, Möbius (of the Möbius band fame) and others, the creator of the subject is without doubt Poncelet.

Poncelet created a revolution in the principles of geometry. He was quite literally a child of the revolution, being born in 1788 France! He seems to have been a staunch supporter of Napoleon. By the way, so was David whose painting I showed you earlier, although he started out as a revolutionary and ended up as a loyalist. Poncelet was on the disastrous Russian campaign and was a prisoner of war at Sarotoff. It was during this imprisonment that he worked out the details of the theory. He enunciated two fundamental principles very clearly. The first one is:

The Principle of Continuity

We all know that parallel lines do not meet. However, let us tilt one of these lines a little this way or that (see *Figure 6*). Now they do meet. As we tilt one of them back to get them to be parallel, the point of intersection moves farther and farther, and so it makes sense to say that they meet at infinity. This is a *principle of continuity.* If you think that it is artificial or farcical, remember the painting that I showed you earlier where parallel lines actually met and you did not object then!

Figure 6 Tilting lines to make them parallel.



But then, should we tilt it clockwise or anti-clockwise? If we think of 2 points at infinity, one on the left and one on the right, it again violates the principle of continuity, since two lines have no business to meet at two points. Therefore we are obliged to assume that the right infinity has to be the same as the left infinity. So we add to a whole

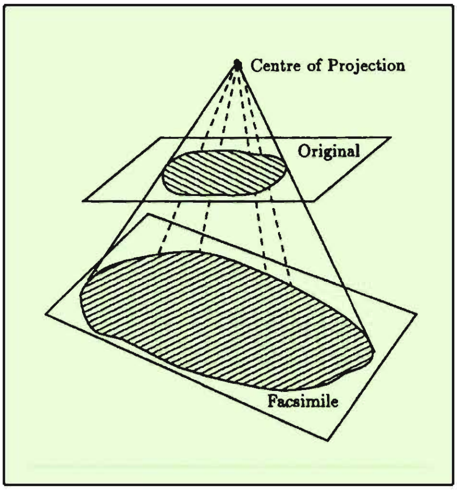


Figure 7 Central projection.

direction, one point at infinity. In other words, we think of the plane as surrounded by points at infinity, one in each direction, except that we identify the two points at infinity which are in diametrically opposite directions. Now we can regard the set of points at infinity also as a line, *the line at infinity.* So one studies geometry in this extended space. Once the space has been extended, we do not give any special status to the line at infinity. It is treated as just another line. Now we may make transformations within this geometry to simplify things if needed. The central projection mentioned above is such a transformation. Here we project things from one plane, the original, into another, the facsimile plane (see *Figure* 7). Note that there is a line on which this projection is not defined, namely the intersection of the plane through the centre of projection, parallel to the facsimile, and the original plane. The reason is that this line becomes the line at infinity in the facsimile. Since we have extended the plane in any case, the map is defined everywhere.

We have ensured that in our geometry any two lines intersect, including parallel lines. How about other geometric figures ?

Let us examine a conic. This is given by a quadratic equation. The line AB intersects the conic at two points but the line CD does not intersect the conic at all (see *Figure 8*). Continuity breaks down once again. How shall we restore it? If complex coordinates are used,

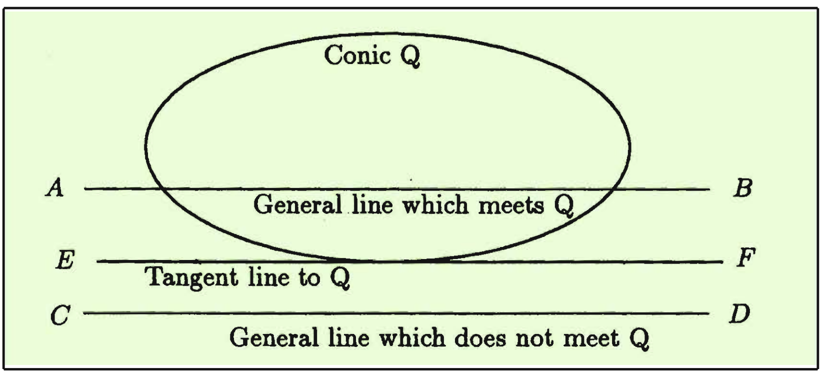


Figure 8 A conic meets a line in different ways.

instead of real ones, then one easily sees that CD intersects the conic at two points, as well. So another consequence of the continuity principle is that it is better to carry out our geometry in the complex world rather than the real world.

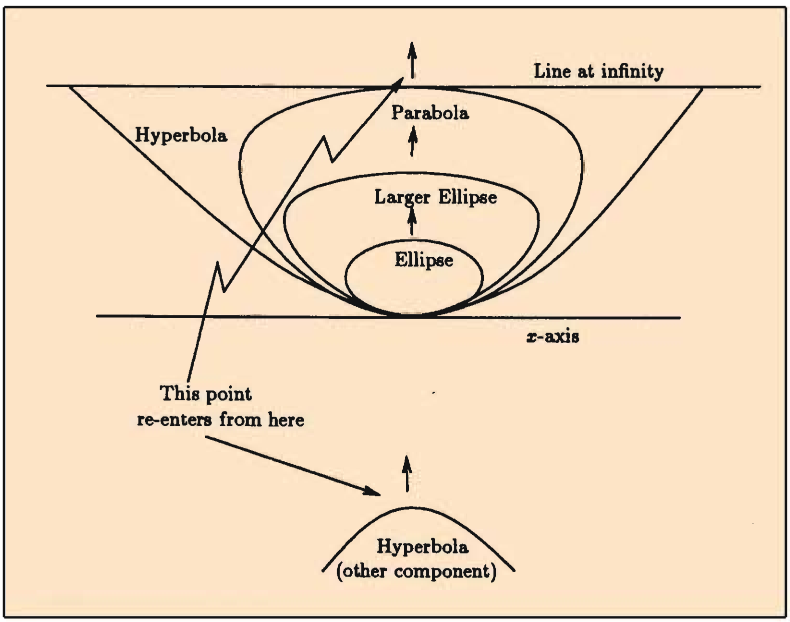
There is one more problem however. If AB touches the conic, then the intersection consists of only one point. By just moving the line a little and making the position a limiting one we see that in some sense this intersection point should be given some weightage or *multiplicity*. This can also be taken care of, thanks to the great French geometer, Grothendieck, of the twentieth century.

Grothendieck postulated that the intersection consists of one point all right, but this point comes with an additional structure, which will detect the fact that it is of multiplicity 2. This kind of animals are called *schemes* and modern algebraic geometry is a menagerie of such animals.

After this, we may say that the intersection of *every* line with the conic, has multiplicity 2.

Again, consider the locus given by $x^2 + t\,y^2 = y$ (see *Figure* 9). An ellipse becomes a parabola in the limit as t becomes smaller. To illustrate this, if we throw a projectile, the locus it traces out is a piece of a parabola. But when a planet is subjected to a central force and is initially moving at a velocity, then it describes an ellipse!

Figure 9 Different conic sections and the line at infinity.



We explain the continuity principle here by saying that when the ellipse touches the line at infinity it is a parabola. However, since the line at infinity has no special status in our geometry, projective geometry does not distinguish between a parabola and an ellipse.

Moreover if we push the parabola a little further, say t is negative in $x^2 + ty^2 = y$, the piece of the parabola begins appearing on the left side, it has now become a hyperbola.

Moral

a) If you need good intersection properties, then the usual space has to be enlarged with some points at infinity (one for each direction, with the understanding that opposite directions are identified).

b) You allow complex coordinates. To each complex line one associates a point at infinity.[1]

c) If we allow the geometric figure to vary in a continuous manner, (the official term is *flat deformation*), the number of intersections, counted with multiplicity, does not change !

Bezout's Theorem. *The intersection of a curve of degree m and a curve of degree n has multiplicity mn.*

The proof is almost trivial after all our preparations. Deform the curves into a union of m lines and n lines respectively. Now the statement is clear. By continuity the statement is true for the original situation as well.

The other important principle set forth by Poncelet is:

The Principle of Duality

Consider the projective space which is obtained by extending our ordinary space by adding some points at infinity. Although this can be done in any number of dimensions, let us stick to the plane. Here we have geometric objects such as empty set, points, lines and the plane itself. Poncelet gave a dictionary, by which he changed the incidence relations between these objects. That is to say, for the word *point* read line and for the word *line* read point. For the phrase *intersection of two lines* read the join of two points, and so on.

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[1] My friend and colleague, R Sridharan drew my attention to the fact that Poncelet himself understood the imaginary points of intersection as an extension of geometric imagination and indeed resisted the introduction of complex co-ordinates.

The way to set this thing up is to think up a dual projective plane, and associate to every point in our projective plane a line in the dual and so on. Thus for every theorem in the first projective plane, we get a dual theorem on the second projective plane, which for this reason is called the *dual* projective plane. A locus of points in one will correspond to the envelope of lines in the other, and so on. I do not have the time to explain this important notion in a systematic way. I content myself by saying that this remarkable realisation has been worked out at various levels: Poincaré duality, Serre duality, Grothendieck duality, Verdier duality, and so on.

The two important principles which I outlined above, particularly after things had been strengthened by the introduction of schemes and flatness, gave rise to Intersection Theory and Duality Theory, which have been at the centre of Algebraic Geometry to the present day.

Dedication

Personally I got my initiation into this subject from our professor T R Raghava Sastri at college. He did not know modern algebraic geometry, but had an excellent geometric intuition and an ability to communicate his enthusiasm to his students. I wish to dedicate this talk to him.

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Errata

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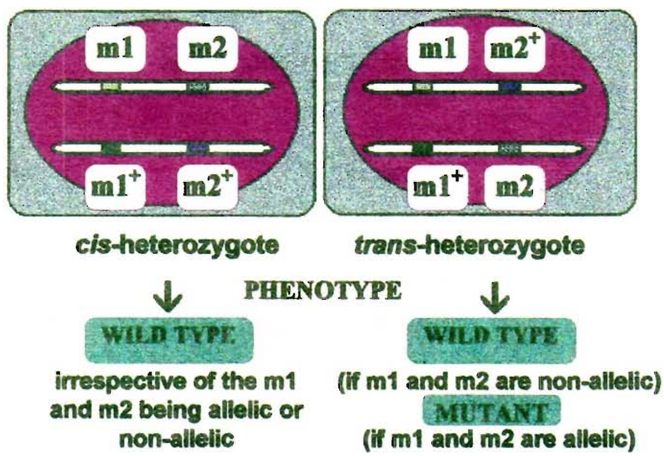
Page 43: Box 3

This relates to the interchange of the terms 'allelic' and 'non-allelic' in the right panel with reference to the wild-type and mutant phenotype. The corrected version of *Box 3* is provided below.

BOX 3

cis-trans **TEST FOR ALLELISM & COMPLEMENTATION**



Conventional genetic test to determine if two linked mutations affecting a given phenotype are allelic or non-allelic compares the phenotype of heterozygous individuals that carry both the mutant genes, (e.g., *m1* and *m2*) in *cis* (i.e., both the mutant genes on one chromosome and both the wild type alleles ($m1^+$ and $m2^+$) on the other homologue, <u>left panel</u>) or in *trans* (i.e., each of the two homologues carries one mutant and one wild type gene, <u>right panel</u>). If the two mutants are *non-allelic* (do not affect the same "gene"), the trans-heterozygote shows wild type phenotype. On the other hand, if they affect the same "gene" (i.e., they are *allelic*), the trans-heterozygote shows a mutant phenotype.