On Shapes of Algebraic Loci [1]

Wild-life in the Forests of Geometry

Subhashis Nag

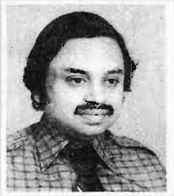
It is of great interest to determine the topological shape of the locus of solutions of any given set of polynomial equations. Answer to this geometrical question leads us to the interaction of topology, algebraic/analytic geometry and differential geometry.

We have all surely studied something about certain simple loci defined by algebraic (namely polynomial) equations in the space of several real variables. Familiar examples are:

- $\{(x,y) \in \mathcal{R}^2 : y=x\}$ defines a straight line — topologically the locus is the real line $\mathcal{R}$.
- $\{(x,y) \in \mathcal{R}^2 : x^2+y^2=1\}$ defines the unit circle, S^1.
- $\{(x,y) \in \mathcal{R}^2 : x^2-y^2=1\}$ defines a hyperbola; this one is, topologically, the disjoint union of two copies of $\mathcal{R}$
- $\{(x,y,z) \in \mathcal{R}^3 : x^2+y^2+z^2=1\}$ defines a 2-dimensional surface — the (unit) sphere, S^2

Notice how just a little change of sign in the polynomial from the second to the third example above completely changes the topological shape of the locus. Whereas, if we were to replace the equation in the second example by say $1000x^2 + 100y^2 = 1$ we would obtain an ellipse which is topologically no different from the circle S^1.

Thus the question arises, given a set of polynomial equations in some real or complex variables, can we determine the topological shape of the locus of solutions? Indeed what shapes can arise? Since solving equations is a fundamental matter in mathematics, the questions at hand are pretty basic. (Note that the solutions form a subset of the appropriate real (or complex) Euclidean



Subhashis Nag received his B.Sc(Hons) from Calcutta University and Ph.D from Cornell University. Subsequently he has worked at many institutions in India and abroad, and is presently a Professor at the Institute of Mathematical Sciences, Chennai. His research interests centre around complex analytic geometry and its intimate relation with mathematical physics via 'string theory'.

[1] Based on an invited talk at the Mathematical Carnival, Indian Institute of Technology, Chennai, March 10,1996.

Given a set of polynomial equations in some real or complex variables, can we determine the topological shape of the locus of solutions?

space, according as we use polynomials with real (complex) variables.)

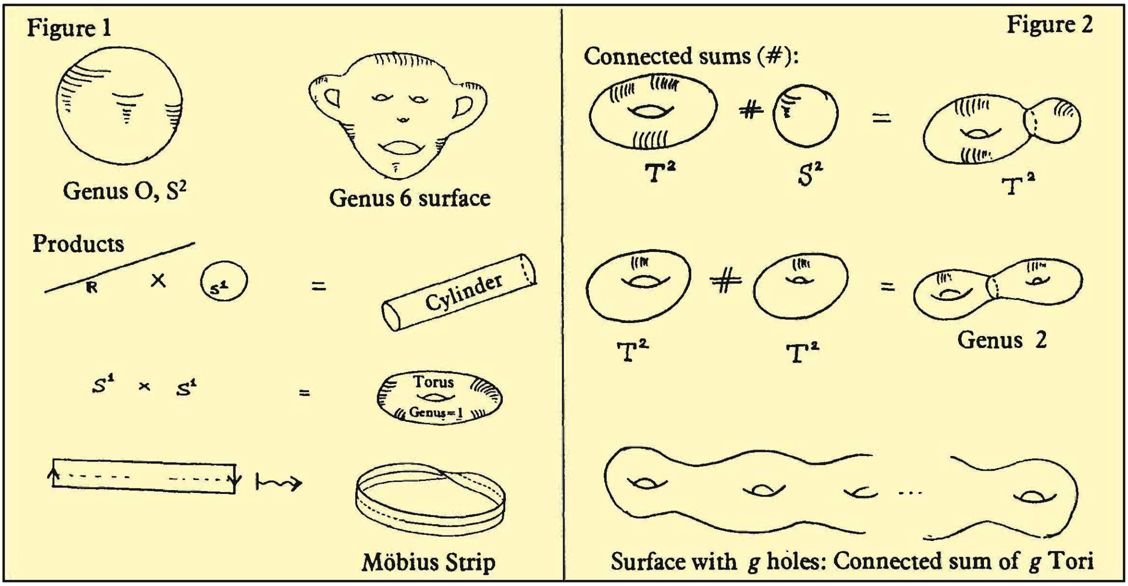
In this article I want to give you just a little hint of how various areas of *Topology, Algebraic/Analytic Geometry and Differential Geometry* interact to answer such geometrical questions about algebraic loci.

We will concentrate on 2-dimensional loci (surfaces) — since these are not very difficult to visualize. In general we will be looking at one real equation in three real variables, or two real equations in four real variables to define the surfaces. Notice that the latter case arises naturally if we take one complex polynomial equation to determine the locus in the space of two complex (i.e., four real) variables. We shall explain these through some examples.

Topology of 'Manifolds'

Firstly then, some topology (='rubber sheet geometry'). We want some list of 'good' topological spaces which could be

Figure 1 (left) and Figure 2 (right)

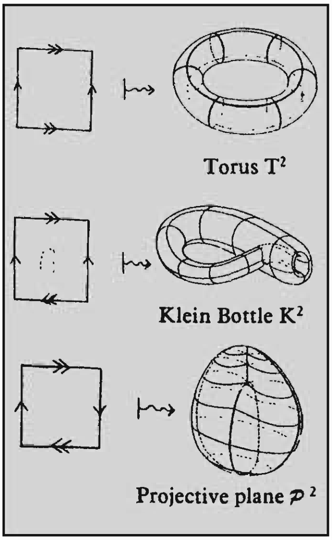


candidates for the shapes of our loci. Therefore we introduce the concept of n-dimensional *manifold* X, as a space which is locally (topologically) equivalent to the n-dimensional Euclidean space $\mathcal{R}^n$. We often indicate the dimension of a manifold by the superscript n. Some examples are shown in the figures.

There are many ways of building up new, and more complicated, manifolds from old ones. ('New lamps for old..!'). We consider the following operations:

- Connected sums — here we cut out little n-dimensional discs from two n-dimensional manifolds and paste the resulting surfaces together along the boundaries of the excised discs. See *Figure 2*. We get a (two-sided) *surface of genus g* by making a connected sum of g copies of the torus T^2.
- Cartesian products – here the dimension of the resulting manifold is the sum of the dimensions of the factors. Thus the cylinder = $\mathcal{R} \times S^1$ as well as the torus $T^2 = S^1 \times S^1$ appear this way. The Möbius strip is a more general type of 'twisted product' (called a fibre bundle) of the real line with the circle. See *Figure 1*.
- Quotient spaces – these are identification spaces obtained by appropriate gluing operations. See *Figure 3* for some interesting examples.

Figure 3



We can obtain the following three compact 2-dimensional manifolds (*Figure 3*), by identifying the two pairs of opposite sides of a square rubber sheet (make either a 'parallel' or 'anti-parallel' identification for each pair).

(i) torus $T^2 = S^1 \times S^1$
(ii) projective plane $\mathcal{P}^2$. This is the 2-sphere S^2 with diametrically opposite points identified. Do you see that from the picture?
(iii) Klein bottle K^2.

The cases (ii) and (iii) have no faithful pictures in three dimensional Euclidean space — so they are a bit difficult to visualize. If you cut out a hole in K^2, then you can faithfully

It is not necessary that an algebraic locus should always look like a manifold.

embed K^2– disc in $\mathcal{R}^3$; see *Figure 3* The figure shown for $\mathcal{P}^2$ is an 'immersion' in $\mathcal{R}^3$.

One-sided (non-orientable) manifolds: Note that the manifolds (ii) and (iii) are 'one-sided' surfaces (like the Möbius strip). (You can paint the entire surface in one colour without ever lifting your paint brush!) Question: Think about whether the connected sums of various copies of (i), (ii) and(iii) amongst themselves will be 1-sided or not.

By making connected sums of these basic objects we get a very interesting list of *compact* (namely, without any boundary and closing up on itself) 2-dimensional manifolds. We get the following complete lists (see Siefert and Threlfall in Suggested Reading):

Orientable (two-sided) compact surfaces: The sphere S^2 (genus = 0), the torus T^2 (genus = 1),..., the g-holed torus (genus $g = 2, 3, 4, \ldots$). It is an exhaustive list of compact two-sided surfaces.

Non-orientable (one-sided) compact surfaces: We, of course, get a whole lot of compact *one-sided* surfaces by performing connected sums of copies of $\mathcal{P}^2$ and K^2. But here again, the entire family of non-orientable compact surfaces is known to arise by taking connected sums of copies of $\mathcal{P}^2$ alone.

Question: Can you see that the Klein bottle K^2 is itself the connected sum of two copies of the projective plane $\mathcal{P}^2$?

Algebraic Manifolds

The upshot of the above foray into the realm of topology is that we see a whole lot of interesting manifolds even when we restrict to two dimensions. We ask whether algebraic loci can *realize* these shapes. (Warning: It is not necessary that an algebraic locus should always look like a manifold. What about $xy = 0$, or curves with nodes, that you must have met in elementary coordinate geometry? The point is that at certain locations algebraic loci can develop 'singularities'. At present, however, we will think only about the 'non-singular' or 'smooth' algebraic loci — i.e., those that represent manifolds.)

We will now work with solutions in *complex* space. Since the complex numbers are algebraically closed, questions about solution sets of polynomial equations become actually *easier* to analyze over $\mathcal{C}$ than over $\mathcal{R}$. (The real locus can be obtained from the corresponding complex locus by taking the intersection with the subspace where the imaginary parts vanish.)

Notice that one complex equation represents a *pair* of real equations (real part = 0, *and* imaginary part = 0). So one certainly expects to get 2-dimensional surfaces as the loci of solutions for one complex polynomial in two complex variables.

The surface determined as the locus of solutions:

$$\mathcal{L} = \{(z, w) \in \mathcal{C}^2 : P(z, w) = 0\}$$

where P is a polynomial in two variables, must be unbounded, and therefore non-compact, in $\mathcal{C}^2$. In fact, by the algebraic completeness of the complex field, for any large complex value of z there must be solutions w satisfying P. We wish to compare the surface $\mathcal{L}$ with the compact manifolds we talked about earlier. Because of the non-compactness of $\mathcal{L}$ it turns out that, in non-singular situations, $\mathcal{L}$ must be topologically identical with one of our list of compact 2-dimensional surfaces minus some holes or punctures.

Examples:

$$\mathcal{L}_1 = \{ (z, w) \in \mathcal{C}^2 : w^2+z^2 = 1\}$$

$$\mathcal{L}_2 = \{ (z, w) \in \mathcal{C}^2 : w^2+z^3 = 1\}$$

$$\mathcal{L}_3 = \{ (z, w) \in \mathcal{C}^2 : w^2+z^5 = 1\}$$

There is a neat and natural way to associate a *compact* surface to any given polynomial locus.

I will not keep the answers a secret from you!

$\mathcal{L}_1$ is S^2 minus 2 points
$\mathcal{L}_2$ is T^2 minus 1 point
$\mathcal{L}_3$ is $T^2 \# T^2$ minus 1 point.

Compactifying the locus: By obvious intuitive process of 'filling in the punctures' it should be clear to you that there is indeed a *compact* surface associated to the locus $\mathcal{L}$ given above. In fact, this intuitive process can be made mathematically rigorous by homogenizing the polynomial P and looking for the locus of solutions of the homogenized creature in *complex projective space*, $\mathcal{CP}^2$, of two complex dimensions, (instead of in $\mathcal{C}^2$). See Walker in Suggested Reading for understanding this. (*Example* : For the second case $\mathcal{L}_2$ above, the homogenized polynomial would be $w^2 t + z^3 - t^3 = 0$, where t is introduced as the 'homogenizing variable'. The locus in $\mathcal{CP}^2$ now indeed becomes the compact (unpunctured!) torus T^2, as you might expect.) Finally then, it transpires that there is a neat and natural way to associate a *compact* surface toany given polynomial locus $\mathcal{L}$ as above. The resulting objects — which surely figure among some of Nature's loveliest gifts to us— are called *compact Riemann surfaces*, or alternatively, *complex algebraic curves*. See Farkas and Kra, Siegel, Walker in Suggested Reading for more details.

If you had powerful computer graphics you could actually try to *plot the locus* of solutions and attempt to see the answers I assert! Can you produce a more general polynomial so that the locus surface is of any desired genus g, given an arbitrary $g \geq 0$!?

Orientability of complex loci: Notice that all the loci seen from the complex equations above are *two-sided* surfaces. In fact, it turns out that since complex equations define loci with 'complex structure' it is not hard to prove that all these loci *must* be two-sided, (i.e., orientable), surfaces. So $\mathcal{P}^2$, K^2, or their connected sums can never appear as these shapes! Thus there are certain restrictions on the topological type of the shapes that can be realized by complex polynomial loci. That is a sample of the type of results we are after.

On the other hand, one-sided manifolds *can* still be realized as *real algebraic loci.* For example:

Projective plane $\mathcal{P}^2$ in $\mathcal{R}^4$: Verify that the mapping

$$q : \text{The unit sphere } S^2 \to \mathcal{R}^4$$
$$(u, v, w) \mapsto (uv, uw, vw, u^2 - v^2)$$

takes equal values only at diametrically opposite pairs of points. It follows that the image locus is nothing other than the projective plane $\mathcal{P}^2$. By eliminating (u, v, w) you should find the algebraic equations in $\mathcal{R}^4$ representing the image. Thus $\mathcal{P}^2$ can be indeed embedded as a real algebraic locus in four dimensional real space!

Differential Geometry on Manifolds

I would like to close with a method from differential geometry – involving the curvatures of surfaces – that can be used to deduce the topological type.

Gaussian curvature of surfaces: Suppose that the locus at hand is a smooth, compact, two-sided surface, X. According to Gauss, we can calculate the *curvature* of X at each point $x \in X$ as follows. Recall that the curvature of a plane curve at a given point on it is the reciprocal of the radius of the circle of closest fit (the osculating = kissing circle) at that point. Now we can look at a whole family of plane curves lying on X and passing through x, by considering the sections of X by normal planes (namely, the planes containing the normal direction at x). (We are thinking of X as embedded in Euclidean space $\mathcal{R}^3$.) As the plane containing the normal rotates in space, one obtains this family of curves, each of which has a certain value for its curvature at x. The curvatures of these curves come with sign.

Gaussian curvature is a real-valued function and is positive at 'mountain-top-type' (or 'lake-bottom-type') points, whereas it takes negative values at 'saddle-type' points.

Indeed, choose a positive direction for the normal vector at x. Then the curves for which the osculating circle has its centre on the positive side of the normal will be assigned positive curvature, whereas the curves for which that circle lies on the negative side will be assigned negative curvature.

Now let k_{max} and k_{min} denote the maximum and minimum values attained by the curvature as the plane of section rotates. Gauss prescribes the product of k_{max} and k_{min} as the curvature of the surface itself at x:

$$\text{curvature}\,(x) = k_{max}\, k_{min}$$

(Notice that the sign of this product is independent of the choice made for the orientation of the normal vector.) Thus, Gaussian curvature is a real-valued function on X; it is positive at 'mountain-top-type' (or 'lake-bottom-type') points, whereas it takes negative values at 'saddle-type' points. (At 'saddle-type' points the curves representing the maximum and minimum values of curvature are concave towards *opposite* sides of X. The two osculating circles lie on opposite sides of the tangent plane at x. On the other hand, at places of positive curvature these two curves are concave towards the *same* side of the surface). It is a remarkable fact that the curves of maximum and minimum curvature, at each point of x, are perpendicular to each other! I recommend the very interesting book by Hilbert and Cohn-Vossen (Suggested Reading) to learn more about the curvature of surfaces.

It is a great discovery of Gauss that the Gaussian curvature we just defined is an 'intrinsic' notion – depending only on the local geometry of X in a neighbourhood of x; it does not really depend on the actual embedding of X in an ambient space, (even though the definition we gave above clearly requires such an embedding).

The Gauss-Bonnet Theorem: Now consider the total integral of the curvature function over the whole space. The Gauss-Bonnet theorem asserts:

$$\int\int_X \text{(curvature)}\, d\,(\text{ area element}) = 4\pi\,(1-g)$$

where g is the genus of X. So this topological quantity – the genus – can be calculated by working out the integral on the left side! To calculate that integral, remember that one is doing calculus (or 'differential geometry') on the given locus. Isn't that fascinating! See Dubrovin, Fomenko and Novikov in Suggested Reading for a discussion.

It is a great discovery of Gauss that the Gaussian curvature we just defined is an 'intrinsic' notion – depending only on the local geometry of X in a neighbourhood of x.

If X is the round sphere of radius r, then the curvature on it is everywhere constant at $1/r^2$. Do you see that the Gauss-Bonnet theorem is satisfied? On a smooth torus ($g=1$), for example, the theorem says that the total contribution from places of positive curvature must exactly balance with that from the places of negative curvature. (Can you have a torus whose curvature is everywhere zero? A 'flat' tyre!?)

A wild-life hunt: All the examples and methods that I have indicated above have vast generalizations organized into comprehensive theories. Manifolds are indeed certain species of the fascinating wild-life roaming in the forests of geometry. The hunting horns are blaring, I wish you a happy hunt!

Suggested Reading

- D Hilbert and S Cohn-Vossen. *Geometry and the Imagination*. Chelsea. New York, 1952.
- C L Siegel. *Topics in Complex Function Theory*. Vol. 1. Wiley-Interscience. New York, 1969.
- R Walker. *Algebraic Curves*. Princeton Univ. Press. 1950. (Republished by Springer-Verlag. New York, 1978).
- H Farkas and I Kra. *Riemann Surfaces*. Springer-Verlag. New York, 1980.
- H Siefert and W Threlfall. *A Textbook of Topology*. Academic Press. New York, 1980.
- B Dubrovin, A Fomenko and S Novikov. *Modern Geometry –Methods and Applications*, Parts 1 and 2. Springer-Verlag. New York, 1984.

Address for Correspondence
Subhashis Nag
The Institute of Mathematical Sciences, CIT Campus, Chennai 600 113, India.
Department of Mathematics University of Southern California, Los Angeles, California 90089-1113, USA.