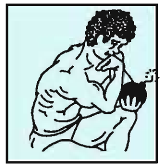
Think It Over



This section of **Resonance** *is meant to raise thought-provoking, interesting, or just plain brain-teasing questions every month, and discuss answers a few months later. Readers are welcome to send in suggestions for such questions, solutions to questions already posed, comments on the solutions discussed in the journal, etc. to* **Resonance** *Indian Academy of Sciences, Bangalore 560 080, with "Think It Over" written on the cover or card to help us sort the correspondence. Due to limitations of space, it may not be possible to use all the material received. However, the coordinators of this section (currently R Nityananda and C S Yogananda) will try and select items which best illustrate various ideas and concepts, for inclusion in this section.*

To Find Four Distinct Positive Integers Such that the Sum of Any Two of them is a Square

Discussion of the question raised in the *Classroom* section (tailpiece) of *Resonance*, Vol.1, No.9, p79.

The problem is to find four distinct positive integers such that any two of them add up to a square. Let a, b, c, d with $a < b < c < d$ be four positive integers which are such that the sum of any two of them is a square. Observing that

$$\begin{aligned} a+b+c+d &= (a+b)+(c+d) \\ a+b+c+d &= (a+c)+(b+d) \\ a+b+c+d &= (a+d)+(b+c), \end{aligned}$$

we need to find a number which can be written as a sum of two non-zero squares in three different ways. We proceed to find such a number.

To begin with, note that if two numbers n, n' can each be expressed as a sum of two squares, nn' can also be so expressed in two ways. Indeed, if

$$n = k^2 + l^2, \; n' = k'^2 + l'^2$$

then

$$\begin{aligned} nn' &= (kk' + ll')^2 + (kl' - lk')^2 \\ &= (kl' + lk')^2 + (kk' - ll')^2. \end{aligned}$$

Start with 25 and 13 both of which are sums of two squares, $25 = 3^2 + 4^2$, $13 = 2^2 + 3^2$. By the identity, $25 \times 13 = 325$ can be expressed as a sum of two squares: $325 = 10^2 + 15^2 = 6^2 + 17^2 = 1^2 + 18^2$. (Note that $10^2+15^2=5^2(2^2+3^2)$).

We show that 13×25^2 has three representations as a sum of two squares and gives us a solution. Consider the following representations (among others):

$$8125 = 30^2 + 85^2 = 50^2 + 75^2 = 58^2 + 69^2.$$

Thus, we take $a+b+c+d = 8125$ and look for solutions a, b, c, d in positive integers. Then $a+b, a+c, a+d, b+c, b+d, c+d$ are precisely $30^2, 85^2, 50^2, 75^2, 58^2, 69^2$ in some order. We have

$$a+b < a+c < a+d < b+d < c+d$$

and

$$a+b < a+c < b+c < b+d < c+d.$$

We arbitrarily take $b+c$ to be less than $a+d$. So we have $a+b = 30^2$, the least of the squares, $a+c = 50^2$, the next smallest, and $c+b = 58^2$. We get $c-b = 1600$ and solving for c, b we have $c = 2482, b = 882$. From this we get $a = 18$, $d = 4743$. Thus $\{18, 882, 2482, 4742\}$ is a set of four positive integers with the required property.

The same method can give large integer solutions too. For example, the following solutions can be obtained by choosing suitable squares.

$\{4190, 10210, 39074, 83426\}$, $\{7070, 29794, 71330, 172706\}$

S H Aravind
12, First Main Road
Ponmeni Jayanagar
Madurai 625010, India