Carl Friedrich Gauss

D Surya Ramana

D Surya Ramana is a junior research fellow at the Institute of Mathematical Sciences, Chennai. He works in the area of analytic number theory.

So varied was the mathematical work of Gauss that this article is limited in scope by the author's own special interests.

Carl Friedrich Gauss, whose mathematical work in the 19th century earned him the title of Prince among Mathematicians ,[1] was born into the family of Gebhard Dietrich Gauss, a bricklayer at the German town of Brunswick, on 30 April 1777.

Gauss was a child prodigy whose calculating prowess showed itself fairly early in his life. Stories of his childhood are full of anecdotes involving his abilities that first amazed his mother, then his teachers at school and much more recently his biographers to the present day. For instance, when he was barely ten years old he surprised his teachers by calculating the sum of the first hundred natural numbers using the fact that they formed an arithmetical progression.

While his talents impressed his mother Dorothea, they caused a great deal of anxiety to his father. Apparently afraid of losing a hand that would otherwise ply some useful trade, he very reluctantly permitted his son to enroll in a gymnasium (high school) at Brunswick. Gauss studied here from 1788–1791.

Perhaps conscious of the fact that his father would not sponsor his studies anymore, Gauss began to look for a patron who would finance his education beyond the gymnasium. Encouraged by his teachers and his mother, Gauss appeared in the court of Duke Carl Wilhelm Ferdinand of Brunswick to demonstrate his computational abilities. The Duke immediately recognised the genius of his subject and granted the fourteen year old Gauss 'whatever means should be needed for the continued training of such a gifted person'.

In 1792 Gauss entered the Collegium Carolinum and also made the first important studies of his mathematical life. These studies, which were on the iterative calculation of arithmetic and geometric means starting with two positive numbers, had already led him to the study of elliptic functions, a subject that would develop much later at the hands of Jacobi and Abel. In the following year, when he was merely fifteen, he had empirically conjectured an asymptotic formula for the number of primes $\pi(n)$ less than a given integer n. This formula, which today is known as *the prime number theorem*, was proved only a century later by de la Vallée-Poussin and Hadamard. However, the most significant contribution of his days at the Collegium Carolinum was the construction of the regular 17 sided polygon using only a ruler and compass. This simply stated problem in elementary geometry, which had remained unsolved since the time of the Greeks, had beneath its surface certain algebraic and number theoretic notions that needed the genius of Gauss to come to light. Indeed, not only had he constructed the regular 17 sided polygon, but also had determined precisely which regular polygons could be constructed by ruler and compass alone.

When he was barely ten years old Gauss surprised his teachers by calculating the sum of the first hundred natural numbers using the fact that they formed an arithmetical progression.

It is said that upto the point when he solved this problem, Gauss had not decided whether he would pursue mathematics or humanities, which was economically more beneficial at his time. His solution to this two thousand year old Greek problem elated him so much that he decided to pursue mathematics.

The fact that he had discovered the principle based on which one "divides a circle into 17 parts and so forth" is also the first entry in his mathematical diary. The entry is dated 30 March 1796. This diary, which he kept throughout his life, contained many of those ideas that had come to him but which he had not been able to work out fully and publish. His notings in this diary, which was discovered among his papers after his death, have been the basis for several investigations even up to recent times.

Simultaneously with noting this discovery in his diary he also announced it in a literary journal. In this announcement he says

Gauss' Construction of the 17-gon

To construct a regular 17-gon geometrically (using ruler and compass) we must find the seventeenth roots of unity and then locate the roots on the unit circle geometrically. The seventeenth roots of unity are found by solving the equation:

$$x^{17} - 1 = 0.$$

The roots are 1, $\cos(2r\pi/17) \pm i\sin(2r\pi/17)$, $r = 1,2,\ldots,8$.
Let $y_r = \cos(2r\pi/17), r = 1,2,\ldots,8$. Then using the fact that the sum of the roots of the equation is zero, we have

$$2(y_1 + y_2 + \ldots + y_8) + 1 = 0 \text{ i.e., } y_1 + y_2 + \ldots + y_8 = -1/2.$$

From the standard conversion formulae we get $2y_iy_j = y_{i+j} + y_{|i-j|}$. Further, pairs of $y_1, y_2, \ldots, y_8$ have a remarkable cyclic property which is the key to the construction:

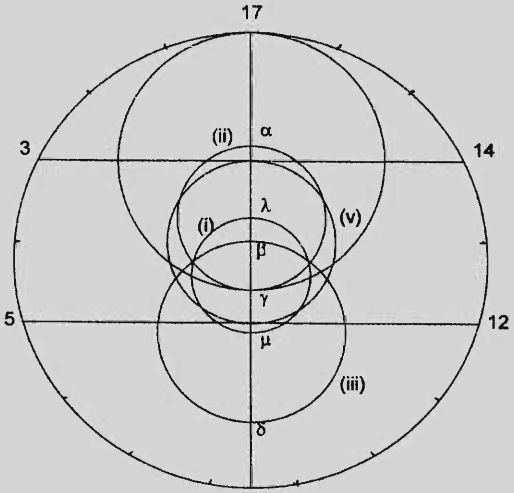
$$2y_1y_4 = y_3 + y_5,\ 2y_3y_5 = y_2 + y_8,\ 2y_2y_8 = y_6 + y_7,\ 2y_6y_7 = y_1 + y_4,$$

Let $y_1 + y_4 = 2\alpha$, $y_3 + y_5 = 2\beta$, $y_2 + y_8 = 2\gamma$, $y_6 + y_7 = 2\delta$. Then,

$$\alpha + \beta + \gamma + \delta = -1/4, \text{ and } \alpha\gamma = (y_1 + y_4)(y_2 + y_8)/4 = (y_1 + y_2 + \ldots + y_8)/8 = -1/16.$$

Similarly, $\beta\delta = -1/16$. Also

$$\alpha\beta = (y_1 + y_4)(y_3 + y_5) = (2y_1 + y_2 + 2y_4 + y_6 + y_7 + y_8) = -1/16 + \alpha/4 - \beta/4.$$



Continued...

Similarly, $16\,\beta\gamma = -1 + 4\,\beta - 4\,\gamma$. So also for $16\,\gamma\delta$ and $16\,\delta\alpha$.

Let $\alpha+\gamma = 2\,\lambda$, $\beta+\delta = 2\,\mu$. Then

$$\lambda + \mu = -1/8,\ 4\,\lambda\mu = \alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha = -1/4.$$

Thus, λ,μ satisfy the quadratic equation: $y^2 + (1/8)y - (1/16) = 0$. The pairs $\{\beta,\delta\},\{\alpha,\gamma\}$ satisfy quadratic equations involving λ, μ and $\{y_6, y_7\}$, $\{y_2, y_8\}$, $\{y_3, y_5\}$, $\{y_1, y_4\}$ satisfy quadratic equations involving α, β, γ, δ.

The roots of a quadratic equation of the type $y^2 - 2\,a\,y + b = 0$ are the ordinates of the points, where the axis $x = 0$ cuts the circle $x^2 + y^2 - 2\,a\,y + b = 0$. Thus we can locate the points y_r, $r = 1, 2, \ldots, 8$, by drawing the corresponding circles.

These steps can be summarised in the following algorithm.
1) Draw a unit circle.
2) Draw a circle centre $(0, -1/16)$ passing through the points $(\pm 1/4, 0)$ cutting the y - axis at λ and μ.
3) Draw two circles centres $(0, \lambda)$ and $(0, \mu)$ passing through the points $(\pm 1/4, 0)$ cutting the y - axis at α, γ and β, δ, respectively.
4) Draw the circle on $(0, 1)$, $(0, \gamma)$ as diameter, cutting the x - axis at $\sqrt{-\gamma}$, 0).
5) Draw the circle with $(0, \beta)$ as centre, passing through ($\sqrt{-\gamma}$, 0).
6) Draw the lines parallel to the x - axis touching this circle .
7) Bisect the intercepted arcs on the unit circle and mark off the vertices.

Renuka Ravindran, Department of Mathematics
Indian Institute of Science, Bangalore 560 012, India.

that the discovery of the ruler and compass construction for the 17 sided polygon "is really only a corollary to a greater theory which will be published soon". This greater theory appeared in his book *Disquitiones Arithmeticae* or *Arithmetical Investigations*, which he published in mid 1801 at the age of 25.

The subject matter of this book is number theory, the branch of mathematics that deals with the properties and relations between integers. In the *Disquitiones*, Gauss developed all the number theory of his time from first principles and in fact went far ahead.

The high point of the initial part of the book, comprising the first four chapters, is the first complete proof of the so called *law*

Gauss proved that a regular polygon with n sides, where n is an odd integer, can be constructed using a ruler and compass alone if and only if n can be expressed as a product of distinct Fermat primes.

of quadratic reciprocity. This law had been conjectured variously by Euler, Lagrange and Legendre, though none of them were able to prove it. Unaware of all this Gauss had independently conjectured the law and shortly before his nineteenth birthday, obtained the first correct proof. Gauss seemed more enamoured of the law than of the proof he obtained. He called it 'the fundamental law' and 'the golden law' and spent many years generalising it. His attempts at this generalisation which was followed up by his disciple Eisenstein and thereafter by other great number theorists of the nineteenth and twentieth centuries, resulted in the creation of an important branch of present day number theory called *Class field theory*.

The second part of his book, comprising the fifth and sixth chapters, dealt with his theory of quadratic forms and its applications. A quadratic form, in this context, is an expression of the type $ax^2 + bxy + cy^2$, where a, b, c are given integers. The principal question at the time of Gauss' investigation was to determine, given an integer m and a quadratic form as above, if there exist integer values x,y such that $ax^2 + bxy + cy^2 = m$.

Like with quadratic reciprocity, Gauss had come to this question on his own. In his study on quadratic forms, which is often considered the best of his published works, he went far ahead of his time. It is this work that led him to suspect the existence of generalisations of the law of quadratic reciprocity.

The seventh and the final chapter of his book was devoted to the development of the theory that Gauss had announced in connection with the construction of the seventeen sided regular polygon. Inaugurating the study of what are now called cyclotomic numbers, Gauss proved that a regular polygon with n sides, where n is an odd integer, can be constructed using a ruler and compass alone if and only if n can be expressed as a product of distinct Fermat primes[2].

[2] A Fermat prime is a prime integer that can be expressed as $2^{2^k}+1$, for some integer k.

By the time this book was published Gauss had also been awarded a doctorate in mathematics. For his doctorate, which was awarded in absentia, by the University of Helmstedt in July 1796, he had submitted a thesis giving a new proof of what is called *the Fundamental Theorem of Algebra.*

However, the event that really brought him recognition was his solution to a problem that was, quite literally, of astronomical proportions.

On Jan 1, 1801, about six months before the publication of *Disquitiones*, the Italian astronomer Joseph Piazzi had sighted the asteroid Ceres. For about forty days he carried out observations on it. In this period it swept an angle of about nine degrees and thereafter disappeared into the light of the Sun.

While the philosopher Hegel ridiculed the astronomers' quest for more planets and asteroids, (he thought he had logically proved that there could be only seven of them), Gauss calmly went about the problem of computing the entire orbit of Ceres with the minimal amount of data that Piazzi had gleaned.

In very little time he had computed the entire orbit and predicted exactly the position at which the asteroid would be seen again. Ceres appeared in the skies exactly as Gauss had said it would and he was immediately recognised a master.

In very little time he had computed the entire orbit and predicted exactly the position at which the asteroid would be seen again. Ceres appeared in the skies exactly as Gauss had said it would and he was immediately recognised a master.

Gauss got married to Johanna Osthoff, the daughter of a tannery owner at Brunswick in 1805. He was still without a permanent job. Despite his fame, his financial position still depended on the stipend from the Duke of Brunswick. To make matters worse, the Duke was engaged in a battle with the then formidable Napoleon Bonaparte whose eventually futile attempt at conquering the world had brought him up to Brunswick. It appeared that the Duke was destined to live longer in the memory of mathematicians rather than on the battlefield. For in October 1806 he was grievously injured and died soon after. His death affected Gauss deeply. However, shortly after this tragedy, the University of Göttingen offered him a professorship

The Gauss-Bonnet theorem states that the total curvature of a surface without boundary is a topological invariant.

of Astronomy which he accepted in November 1806. He remained in that position for the rest of his life.

Matrimony was quick to bestow its favours on Gauss.In 1806 a son was born to him and two years later a daughter. But then tragedy struck again. In 1809 after a third delivery his wife passed away and in a few months their infant son also died leaving Gauss very depressed. In the following year Gauss married Minna Waldeck, the closest friend of his late wife in her last years and by 1816 he was a happily settled man with an awesome reputation, a permanent position and a family with five children, three from his second marriage.

Until 1816 owing to his studies in astronomy Gauss investigated the estimation of errors in observation. In the process, he developed the method of least squares, the law of normal distribution (which is also called Gaussian distribution) and also perturbation methods for studying planetary orbits.

Starting in 1820 he turned to differential geometry of surfaces motivated by his work on geodesy. In 1827 he published his results in a paper that has become a landmark in the subject. In this paper he introduced the notion of what is now called the Gaussian curvature of a surface. In an earlier work on this subject, Euler had introduced two curvatures for a surface embedded in space. However, the Eulerian curvatures depended on the way in which the surface was embedded in space. On the other hand, Gaussian curvature does not change as long as one deforms the surface without stretching or tearing. This is a consequence of Gauss' *Theorema Egregium* (outstanding theorem) that was proved in this paper. He proved that the sum of the angles of a triangle on a surface differs from 180° by an amount proportional to the total curvature of the triangle. This later led to the Gauss-Bonnet theorem which states that the total curvature of a surface without boundary is a topological invariant. Indeed some feel that this paper shows that Gauss had a clear notion of non-Euclidean geometry far ahead of Bolyai and Lobachevsky. A more practical consequence of Gauss' work was a proof of the impossibility of making constant scale maps of the Earth.

Indeed some feel that Gauss had a clear notion of non-Euclidean geometry far ahead of Bolyai and Lobachevsky.

By the year 1831 his varied interests had taken him to Physics. Upon his recommendation the experimental physicist Wilhelm Weber was appointed a professor at Göttingen. Gauss and Weber formed an unbeatable combination. By the year 1833 they had invented the electromagnetic telegraph and built a theory for explaining terrestrial magnetism. Indeed they had also investigated the question of determining the strength of the earth's magnetic field. It is in honour of these studies that the SI unit for magnetic flux density is called a *weber* and one ten thousandth of 1 weber, which roughly corresponds to the terrestrial magnetic flux, is called a *gauss*. During these years (1833–1855), Gauss also worked on potential theory and optics.

Gauss passed away on 23 February, 1855, at the age of 78 — a rather long life for a genius of his calibre and times. Whether the Gods loved him or not, he had surely earned the respect of mathematicians and scientists of his times and those that followed including, by now, perhaps the reader.

Address for Correspondence
D Surya Ramana
Institute of Mathematical Sciences
CIT Campus, Taramani
Chennai 600 113, India

In Brunswick, Germany, in 1780, a stonemason was calculating the wages due to his workmen at the end of the week. Watching was his three-year-old son. "Father," said the child, "the reckoning is wrong." The boy gave a different total which, to everyone's surprise, was correct. No one had taught the lad any arithmetic. The father had hoped his son would become a bricklayer, but thanks to his mother's encouragement, the boy, Carl Friedrich Gauss, became one of the greatest mathematicians in history.