# Nīlakantha, Euler and π

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## Introduction

In this article, we present brief expository accounts of two well-known series for $\pi$: the *Gregory-Nīlakaṇṭha series*,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots \tag{1}$$

also known as *Leibnitz's series*, and the series $\sum_n 1/n^2$, whose sum was found by Leonhard Euler in 1734:

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots \tag{2}$$

The first series is usually known as the *Gregory-Leibnitz* series but recent findings have shown that it was known to mathematicians in Kerala well before Gregory and Leibnitz or the invention of the calculus. Evidently the discoveries were made independent of one another.

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## Preliminaries on Infinite Series

Infinite series are fun to play with—often very pretty identities emerge from their study. Sometimes they are easy to evaluate. We have, for instance, $1/2+1/4+1/8+1/16+\cdots = 1$, and more generally,

$$x + x^2 + x^3 + x^4 + \quad = \frac{x}{1-x} \quad \text{for } |x| < 1. \tag{3}$$

Geometric series of this type are well known at the high-school level, and we shall not say more about them.[1] Another class of series that are easy to evaluate are the ones that yield

[1] The well known paradox of Achilles and the tortoise is resolved by studying the underlying geometric series.

to partial fraction decomposition followed by telescoping. For instance, the series $1/2 + 1/6 + 1/12 + 1/20 + 1/30 + 1/42 +$ is summed as shown below:

$$\begin{aligned} \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots &= \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots \\ &= 1, \end{aligned} \quad (4)$$

after a wholesale cancellation of fractions. More challenging is the evaluation of the series $\sum_n 1/n(n+1)(n+2)$. A partial fraction decomposition yields:

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2(n+2)} - \frac{1}{n+1} + \frac{1}{2n} \quad (5)$$

and a rearrangement of terms gives

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}. \quad (6)$$

(Comment: We must mention here that rearranging the terms of an infinite convergent series in order to compute its sum is not always a valid step. For instance, the series $1 - 1/2 + 1/3 - 1/4 + \cdots$ converges, yielding a sum of $\ln 2$, but by rearranging the terms in different ways we get different sums. For instance, when rearranged as shown below,

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots,$$

with two positive terms followed by one negative term, the series converges to $\ln 2\sqrt{2}$, and the series

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{2} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{4} +$$

with three positive terms followed by one negative term, converges to $\ln 2\sqrt{3}$. The notion of *absolute convergence* helps to make sense of these facts. Suppose that the series $\sum_i a_i$ converges. Then the series converges 'absolutely' if $\sum_i |a_i|$ converges. A theorem of Riemann's (extremely surprising at first encounter) states that if $\sum_i a_i$ converges but not absolutely (that is, $\sum_i |a_i|$ diverges), then by suitably rearranging the terms we can get the resulting series to converge to any desired number whatever! If the $a_i$ are all of one sign, then the notions of convergence and absolute convergence coincide. In the example at hand, rearrangement can be avoided by observing that

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right)$$

Use of the telescoping principle now yields (6).)

Another pretty result is provided by the summation

$$x + 2x^2 + 3x^3 + 4x^4 + \cdots = \frac{x}{(1-x)^2} \tag{7}$$

which holds whenever $|x| < 1$.

However a very different proposition is the alternating series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots. \tag{8}$$

It is not hard to show that the series converges, for by combining pairs of terms it can be written as

$$2\left(\frac{1}{3} + \frac{1}{35} + \frac{1}{99} + \cdots\right) = 2\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}. \tag{9}$$

Since $4n^2 - 1 > n(n-1)$ for $n \geq 1$, it follows that

$$\sum_{n\geq 1} \frac{1}{4n^2-1} < \frac{1}{3} + \sum_{n\geq 2} \frac{1}{n(n-1)} = \frac{4}{3}.$$

Thus the series has a well-defined sum. What is the sum? The 'school-boy' evaluation (very likely the method used by Gregory) proceeds thus: since

$$1 - x^2 + x^4 - x^6 + x^8 - \cdots = \frac{1}{1+x^2} \text{ for } |x| < 1, \tag{10}$$

A theorem of Riemann's (extremely surprising at first encounter) states that if $\Sigma_i a_i$ converges but not absolutely (that is, $\Sigma_i |a_i|$ diverges), then by suitably rearranging the terms we can get the resulting series to converge to any desired number whatever!

we deduce, by integration between 0 and $t$ ($|t| < 1$), that

$$\tan^{-1} t = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \cdots \tag{11}$$

Substituting the value $t = 1$ in (11), we obtain the required sum as $\pi/4$. However the question of convergence is left open in this summation (an infinite series cannot always be integrated term-by-term in this fashion), and we need to fill in the necessary details. This is accomplished as follows. We consider the algebraic identity (note that this is *not* an infinite sum)

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \quad + (-1)^n x^{2n} + (-1)^{n+1} \frac{x^{2n+2}}{1+x^2}. \tag{12}$$

Integrating both sides between 0 and 1, we obtain

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \quad + (-1)^n \frac{1}{2n+1} + (-1)^{n+1} \int_0^1 \frac{x^{2n+2}}{1+x^2}\, dx. \tag{13}$$

The integral on the right side tends to 0 as $n \to \infty$, because

$$\left| \int_0^1 \frac{x^{2n+2}}{1+x^2}\, dx \right| < \int_0^1 x^{2n+2}\, dx = \frac{1}{2n+3}, \tag{14}$$

which tends to 0 as $n \to \infty$. The result follows.

Another approach is via evaluation of the integral $I_n$ defined by

$$I_n = \int_0^{\pi/4} \tan^n x\, dx \tag{15}$$

for positive integers $n$. (This integral is routinely encountered while studying integration by parts). We claim that $I_n \to 0$ as $n \to \infty$. For proof we argue as follows. Obviously $I_n > 0$. Next, the tangent function is convex over the open interval $(0, \pi/4)$, so the chord connecting the points $(0,0)$ and $(\pi/4, 1)$ lies entirely above the tangent curve. The equation of this line is $y = (4/\pi)x$, so it follows that

$\tan x < (4/\pi)x$ for all $x$ in the open interval $(0, \pi/4)$. Therefore

$$0 < I_n = \int_0^{\pi/4} \tan^n x\,dx < \left(\frac{4}{\pi}\right)^n \int_0^{\pi/4} x^n\,dx$$
$$= \left(\frac{4}{\pi}\right)^n \left(\frac{\pi}{4}\right)^{n+1} \frac{1}{n+1}$$
$$= \frac{\pi}{4(n+1)}, \qquad (16)$$

and the result follows (let $n \to \infty$). Next, integration by parts yields

$$I_n = \frac{1}{n-1} - I_{n-2}. \qquad (17)$$

From (17) and the fact that $I_0 = \pi/4$, we deduce that

$$1 - \frac{1}{3} + \frac{1}{5} - \quad + (-1)^n \frac{1}{2n-1} = \frac{\pi}{4} + (-1)^n I_{2n}. \qquad (18)$$

Letting $n \to \infty$, we obtain the Gregory-Nīlakaṇṭha series. This approach yields an unexpected bonus: (since $I_1 = \ln \sec \pi/4 = (1/2)\ln 2$, we obtain

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots. \qquad (19)$$

(**Exercise**: Prove (19).)

## The Gregory-Nīlakaṇṭha Series

Both proofs presented above make use of calculus, and it would seem difficult to arrive at the stated result without using integration. However the Kerala school of mathematics that flourished in the fifteenth and sixteenth centuries seems to have been in possession of just such a proof, due originally to the mathematician Nīlakaṇṭha[2] (1444–1550). We now present this elegant proof. In retrospect it has a certain naturalness to it, for $\pi$ *is* after all connected to the circumference of a circle. Though calculus is not used at any stage in the proof, the intuitive notion of limit is clearly present in the author's mind.

[2] Or so we think ! The facts are unfortunately not too well known. In the text *Yuktibhāsā* , the series is credited to Mādhava of Sangamagrāma (1350-1410), who lived almost a full century before Nīlakaṇṭha The result is stated as follows: if $c$ is the circumference of a circle of diameter $d$, then

$$c = 4d - \frac{4d}{3} + \frac{4d}{5} - \frac{4d}{7} + \frac{4d}{9} - \cdots.$$

See S Balachandra Rao in Suggested Reading for further details.

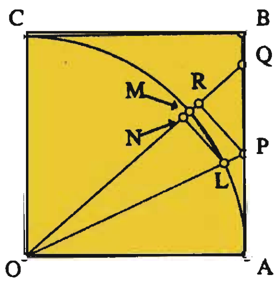


*Figure 1*

In *Figure 1* $OABC$ is a square of side 1, and $AC$ is a circular arc of unit radius. The length of the arc $AC$ is thus $\pi/2$. Let $P$, $Q$ be two points on the side $AB$, the points occurring in the order shown ($A$, $P$, $Q$, $B$), with $P$, $Q$ very close to one another, i.e., with $PQ \ll 1$. Let $OP$ and $OQ$ meet the arc at $L$ and $M$, respectively, and let $PR$ and $LN$ be the perpendiculars from $P$ and $L$ upon line $OQ$. Then $\Delta OLN \sim \Delta OPR$ and $\Delta QRP \sim \Delta QAO$. Therefore

$$\frac{LN}{OL} = \frac{PR}{OP}, \quad \text{and} \quad \frac{PR}{PQ} = \frac{OA}{OQ}. \tag{20}$$

These equations give, since $OL = OA = 1$

$$LN = \frac{PQ}{OP \times OQ}. \tag{21}$$

Now $OP \times OQ \approx OQ^2 = 1 + AQ^2$, and arc $LM \approx LN$. Therefore

$$\text{arc } LM \approx \frac{PQ}{1 + AQ^2}. \tag{22}$$

Now let the side $AB$ be divided into $n$ segments of length $1/n$ each, where $n$ is large, and let the points of subdivision be $P_1$, $P_2$, $\ldots$ $P_{n-1}$; let $P_0 = A$, $P_n = B$. Then $P_iP_{i+1} = 1/n$ for $i = 0, 1, 2, \ldots n-1$. Summing $n$ relations of the type (22) we obtain:

$$\frac{\pi}{4} \approx \sum_{i=1}^{n} \frac{1/n}{1 + i^2/n^2}. \tag{23}$$

(*Exercise*: Fill in the details of the proof of this formula.)

It follows that

$$\frac{\pi}{4} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1/n}{1 + i^2/n^2}. \tag{24}$$

(The discerning reader will note at this point that we have arrived in an unexpected fashion at the 'discretized' version of the integral $\int_0^1 dx/(1+x^2)$ !) The limit is evaluated as follows. We write the fraction $(1/n)/(1+i^2/n^2)$ as the sum of an infinite series, using the fact that $1/(1+x^2) = 1 - x^2 + x^4 - x^6 + \ldots$ for $|x| < 1$:

$$\frac{1/n}{1 + i^2/n^2} = \frac{1}{n}\left(1 - \frac{i^2}{n^2} + \frac{i^4}{n^4} - \frac{i^6}{n^6} + \ldots\right) \tag{25}$$

Therefore

$$\begin{aligned}\frac{\pi}{4} &= \lim_{n\to\infty}\frac{1}{n}\left(\sum_{i=1}^{n}1-\sum_{i=1}^{n}\frac{i^2}{n^2}+\sum_{i=1}^{n}\frac{i^4}{n^4}-\sum_{i=1}^{n}\frac{i^6}{n^6}+\cdots\right)\\ &= \lim_{n\to\infty}\left(1-\frac{\sum i^2}{n^3}+\frac{\sum i^4}{n^5}-\frac{\sum i^6}{n^7}+\cdots\right) \qquad (26)\end{aligned}$$

all summations in (26) from $i = 1$ to $i = n$.

Nīlakaṇṭha observed that the limits of the individual terms in this expression were easy to evaluate. The modern evaluation of $\lim_{n\to\infty}\left(\sum i^2\right)/n^3$ proceeds thus: since $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$, the required limit is

$$\lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{3}+\lim_{n\to\infty}\left(\frac{1}{2n}+\frac{1}{6n^2}\right) = \frac{1}{3}. \qquad (27)$$

It is clear that Nīlakaṇṭha was aware of this as well as each of the following limits (though the language of limits was non-existent at that time):

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n} i^3}{n^4} = \frac{1}{4}, \quad \lim_{n\to\infty}\frac{\sum_{i=1}^{n} i^4}{n^5} = \frac{1}{5}, \qquad (28)$$

and more generally, for any positive integer $k$,

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n} i^k}{n^{k+1}} = \frac{1}{k+1}. \qquad (29)$$

The Nīlakaṇṭha-Gregory series,

$$\frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+ \qquad (30)$$

now follows immediately.

If we study the proofs of Gregory and Nīlakaṇṭha carefully, we see that they both depend upon the expansion of $1/(1+x^2)$ as an infinite series.

## Euler's Summation Of $\sum 1/n^2$

Another series that arises naturally in many settings is the series $\sum_{n=1}^{\infty} 1/n^2$. It is easy to show that the series converges; for $n^2 > n(n-1)$, so $1/n^2 < 1/(n-1) - 1/n$. Therefore for any positive integer $k > 1$,

$$\sum_{n=1}^{k} \frac{1}{n^2} = 1 + \sum_{n=2}^{k} \frac{1}{n^2} \; < \; 1 + \sum_{n=2}^{k} \left( \frac{1}{n-1} - \frac{1}{n} \right)$$
$$= 1 + \left(1 - \frac{1}{k}\right) < 2, \tag{31}$$

What is the value of $\sum_n 1/n^2$? The question remained a mystery and Jakob Bernoulli expressed the feelings of his contemporaries when he wrote: "...If anyone finds and communicates to us that which till now has eluded our efforts, great will be our gratitude..." In 1734 Euler produced a solution using a marvellous piece of reasoning.

that is, the sum $\sum_{n=1}^{k} 1/n^2$ is less than 2. Since the partial sums of $\sum_n 1/n^2$ form a monotone increasing sequence and are bounded above by 2, they must possess a limit. This was quite clear to the contemporaries of Newton, Leibnitz and the Bernoulli brothers. But just what is the value of this limit? A numerical evaluation is easy to do and the sum can be obtained to any desired degree of accuracy if one is willing to do some computation, but can the series be evaluated in 'closed form'? The question remained a mystery and Jakob Bernoulli expressed the feelings of his contemporaries when he wrote: ". . If anyone finds and communicates to us that which till now has eluded our efforts, great will be our gratitude . ". In 1734 Euler produced a solution using a marvellous piece of reasoning. The solution is a gem, and produces a truly unexpected result, but it has some questionable aspects to it, and would fall short of today's standards of rigour.

Euler started with the power series for $\sin x$:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{32}$$

This result was of course available to Leibnitz and his contemporaries; we refer to it today as the *Maclaurin series* for $\sin x$. From this it follows that for $x > 0$,

$$\frac{\sin\sqrt{x}}{\sqrt{x}} = 1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + \cdots \tag{33}$$

Next, Euler made the following observation: if $f(x)$ is a polynomial with roots $\alpha, \beta, \gamma, \ldots$, such that $f(0) = 1$, then

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \cdots = -f'(0). \tag{34}$$

The observation is easy to justify, for the hypotheses tell us that

$$f(x) = \left(1 - \frac{x}{\alpha}\right)\left(1 - \frac{x}{\beta}\right)\left(1 - \frac{x}{\gamma}\right) \cdots \tag{35}$$

Expanding the right side we obtain

$$f(x) = 1 - \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \cdots\right)x + \tag{36}$$

and the stated claim follows. (Actually Euler did not write $f'(0)$ on the right side of (34); rather he claimed that $\sum 1/\alpha = -1\times$ the coefficient of $x$ in $f(x)$. Since the coefficient of $x$ in any polynomial $f(x)$ is equal to $f'(0)$, the two statements are equivalent to one another.) Euler's claim can be stated in another form: Let $f(x)$ be a polynomial with roots $\alpha$, $\beta$, $\gamma$, none of which is 0. Then

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \quad = -\frac{f'(0)}{f(0)}. \tag{37}$$

In this form the proof is still easier: the hypothesis tells us that

$$f(x) = k(x-\alpha)(x-\beta)(x-\gamma) \tag{38}$$

for some constant $k$. Taking logarithms on both sides, we obtain

$$\ln f(x) = \ln k + \ln(x-\alpha) + \ln(x-\beta) + \ln(x-\gamma) + \cdots \tag{39}$$

Differentiating both sides of (39) we obtain

$$\frac{f'(x)}{f(x)} = \frac{1}{x-\alpha} + \frac{1}{x-\beta} + \frac{1}{x-\gamma} + \cdots, \tag{40}$$

and the substitution $x = 0$ in (40) yields (37).

Euler then took a rather large leap: *he considered the function $f(x) = \sin(\sqrt{x})/\sqrt{x}$ to be a 'polynomial of infinite degree'*, namely that given by (33). This 'polynomial' certainly satisfies the relation $f(0) = 1$, because

$$\lim_{x\to 0^+} \frac{\sin\sqrt{x}}{\sqrt{x}} = 1. \tag{41}$$

(Euler did not use the terminology of limits—the notion of limit was relatively undeveloped in his time, and later Cauchy set the notion on firm ground.)

Now what are the roots of the 'polynomial' $f(x)$? Since the roots of $\sin x$ are $n\pi$, where $n$ takes all possible integral values, the roots of $\sin\sqrt{x}$ are $n^2\pi^2$, with $n$ taking all possible integral values. Therefore the roots of the equation $f(x) = 0$ are just the values $n^2\pi^2$ $(n \neq 0)$.

Finally, what is $f'(0)$ for this infinite degree polynomial? Since we already have the Maclaurin expansion of $f(x)$, namely

$$f(x) = 1 - \frac{x}{6} + \frac{x^2}{120} - \frac{x^3}{5040} + \cdots, \tag{42}$$

we see that $f'(0) = -1/6$.

Combining all these deductions into one, we see that

$$\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} = \frac{1}{6}, \tag{43}$$

and so

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{44}$$

The required sum has been obtained—its value is $\pi^2/6$. What a miraculous achievement!

Euler's method yields other such results. Following the comments made earlier, we proceed to write $f(x)$ as an infinite product:

$$f(x) = \left(1 - \frac{x}{\pi^2}\right)\left(1 - \frac{x}{4\pi^2}\right)\left(1 - \frac{x}{9\pi^2}\right) \cdot, \tag{45}$$

that is,

$$\frac{\sin\sqrt{x}}{\sqrt{x}} = \prod_{n=1}^{\infty}\left(1 - \frac{x}{n^2\pi^2}\right) \tag{46}$$

Now the coefficient of $x^2$ in the product

$$\left(1 - \frac{x}{\alpha}\right)\left(1 - \frac{x}{\beta}\right)\left(1 - \frac{x}{\gamma}\right)\cdot\cdot \tag{47}$$

is $\sum 1/\alpha\beta$, which can be written as

$$\frac{1}{2}\left(\left(\sum \frac{1}{\alpha}\right)^2 - \left(\sum \frac{1}{\alpha^2}\right)\right) \tag{48}$$

the summation being over the roots of $f(x) = 0$. Since the roots are $n^2\pi^2$ $(n \neq 0)$, we see that

$$\left(\sum \frac{1}{\alpha}\right)^2 = \left(\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2}\right)^2 = \frac{1}{6^2} = \frac{1}{36}, \tag{49}$$

and

$$\left(\sum \frac{1}{\alpha^2}\right) = \left(\frac{1}{\pi^4}\right)\left(\sum_{n=1}^{\infty} \frac{1}{n^4}\right) \tag{50}$$

Finally, the coefficient of $x^2$ in $f(x)$ (remember that $f(x)$ is still being thought of as a polynomial!) is $f''(0)/2$. To evaluate $f''(0)$ we use L'Hopital's rule. Differentiating the function $f$ given by $f(x) = \sin\sqrt{x}/\sqrt{x}$ twice in succession, we find that

$$f''(x) = \frac{1}{4x^2}\left(\frac{(3-x)\sin\sqrt{x}}{\sqrt{x}} - 3\cos\sqrt{x}\right) \tag{51}$$

Letting $x$ tend to 0, we obtain $f''(0) = 1/60$ (left as an exercise for the reader!). Therefore

$$\frac{1}{2}\left(\frac{1}{36} - \frac{1}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}\right) = \frac{f''(0)}{2} = \frac{1}{120}, \tag{52}$$

and this yields the beautiful result

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}. \tag{53}$$

A few additional remarks can be made here. Euler could equally have used the function $\cos\sqrt{x}$ instead of $\sin\sqrt{x}/\sqrt{x}$. For, writing $f(x) = \cos\sqrt{x}$, we find that the roots of $f(x) = 0$ are $(2n-1)^2\pi^2/4$ where $n$ takes all possible positive integral values. Since $f(0) = 1$, this gives us the representation of $f(x)$ as an infinite product:

$$\cos\sqrt{x} = \prod_{n=1}^{\infty}\left(1 - \frac{4x}{(2n-1)^2\pi^2}\right) \tag{54}$$

On the other hand, $\cos x = 1 - x^2/2! + x^4/4! + x^6/6! - \cdots$, so for $x > 0$,

$$\cos\sqrt{x} = 1 - \frac{x}{2} + \frac{x^2}{24} - \frac{x^3}{720} + \cdots \qquad (55)$$

Arguing as we did earlier (the reader is invited to fill in the missing steps), we find that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}. \qquad (56)$$

Finally, it is trivial to show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \left(1 - \frac{1}{2^2}\right)\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right) \qquad (57)$$

and so the result that $\sum_n 1/n^2 = \pi^2/6$ follows.

Rather remarkably, Euler's method can be used to find the sum of the Gregory-Nīlakaṇṭha series as well. (This did not escape Euler's attention—but then few things did!) We consider the function $f(x) = 1 - \sin x$, for which $f(0) = 1$, $f'(0) = -1$, $f'(0)/f(0) = -1$. The roots of the equation $f(x) = 0$ are $x = 2n\pi + \pi/2 = (4n+1)\pi/2$, where $n$ takes all possible integral values. Moreover, these roots are all *double* roots, as a glance at the associated graph will show. (The $x$-axis is tangent to the graph of $y = 1 - \sin x$ at these values.) We therefore obtain, using Euler's rule:

$$1 = -2\left(\sum_{n \in \mathbf{Z}} \frac{2}{(4n+1)\pi}\right) \qquad (58)$$

(Here $\mathbf{Z}$ is the set of integers, i.e., $\{0, \pm 1, \pm 2, \pm 3, \ldots\}$.) Writing the right side in expanded form and multiplying both sides by $\pi/4$, we obtain the series of Gregory and Nīlakaṇṭha!

Euler used such methods to evaluate the value of the zeta function $\zeta(k)$ for many positive even integers $k$. (The zeta function is defined by

$$\zeta(t) = 1 + \frac{1}{2^t} + \frac{1}{3^t} + \quad = \sum_{n=1}^{\infty} \frac{1}{n^t} \qquad (59)$$

where $t$ is allowed to take complex values as well.) It is known now that $\zeta(k)$ is a rational multiple of $\pi^k$ whenever $k$ is a positive even integer. Interestingly, comparatively little is known about $\zeta(k)$ when $k$ is odd, not even whether the values taken are irrational! (It *is* known that $\zeta(3)$ is irrational—but this is an extremely recent discovery.)

As a tailpiece, we leave the following assertion for the reader to ponder over and to exercise some Eulerian ingenuity. Let the real positive roots of the equation $\tan x = x$ be $\lambda_1, \lambda_2, \lambda_3, \ldots$ Then

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i^2} = \frac{1}{10}. \tag{60}$$

(Warning!: Unlike the sine and cosine functions, the tangent function takes infinite values. How does one get around this?)

It is clear that the proofs of the Gregory–Nīlakaṇṭha series are complete in every respect, while Nīlakaṇṭha's proof is only slightly incomplete. The missing steps are very minor and can easily be filled in, without much effort. Considering the era in which the proof was written — a time when the notion of limit was non-existent elsewhere in the world — this is a remarkable achievement on the author's part.

## Concluding Remarks

What are we to make of the various proofs presented above? In general one can say that there are two kinds of proofs. There are proofs that are complete in every respect, with every minor detail attended to; Euclid's proofs of the infinitude of the primes and of the irrationality of $\sqrt{2}$ (both of which are to be found in *The Elements*) are certainly of this variety. Then there are proofs with a few steps missing which can however be inferred from the main body of the proof. *In particular there is no such thing as an unrigorous proof.* An unrigorous proof is no proof at all!—it is only a suggestion of a proof, and considerable work needs to be done before it can be referred to as a proof. How do the above presentations measure up in the light of these remarks?

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However when we come to Euler's evaluation of $\Sigma_n 1/n^2$, we have quite a different situation before us. While we can sit back and marvel at the sheer virtuosity and brilliance of Euler's work, at its freshness and vitality, we are at the same time forced to admit that it is far from being a proof.

*passant*, one wonders what happened to the very remarkable school of thinkers to which Mādhava and Nīlakaṇṭha belonged: in what directions did they branch out? What kinds of questions did they occupy themselves with?)

However when we come to Euler's evaluation of $\sum_n 1/n^2$, we have quite a different situation before us. While we can sit back and marvel at the sheer virtuosity and brilliance of Euler's work, at its freshness and vitality, we are at the same time forced to admit that it is far from being a proof. To what extent can we apply a result to a power series when all we know is that it holds for polynomials?—that is, when can we express a function $x$ as a product of linear expressions in $x$, using only the roots of $f(x)$ to go by? The answer is: not always. Euler was certainly lucky that his audacious attempt worked; as they say, first time lucky! In general, it is *not* true that if $f$ is a complex-valued function with no singularities and with roots $\alpha_1, \alpha_2, \alpha_3,$ , none of which is 0, then

$$f(x) = f(0)\left(1-\frac{x}{\alpha_1}\right)\left(1-\frac{x}{\alpha_2}\right)\left(1-\frac{x}{\alpha_3}\right)\cdots \qquad (61)$$

Here is a simple counterexample. Consider the function $f(x)$ defined on the complex plane thus: $f(0) = 1$, and $f(x) = (e^x - 1)/x$ for $x \neq 0$. Then $f$ is continuous and differentiable over the whole plane, and its roots are the values $2n\pi i$, where $i$ is $\sqrt{-1}$ and $n$ takes all possible non-zero integral values. We may now be led to infer (after combining pairs of factors) that

$$\frac{e^x - 1}{x} = \prod_{n=1}^{\infty}\left(1 + \frac{x^2}{4n^2\pi^2}\right) \qquad (62)$$

but this is absurd, because the function represented by the expression on the right side is an even function of $x$, whereas the function on the left side is not. (In the correct representation, an additional factor, namely $e^{x/2}$, has to be included on the right side of (62).)

In general there are infinitely many different complex-valued functions that share the same set of roots (with regard to position as well as order); thus, the functions $f(x)$, $e^x f(x)$, $e^{x^2} f(x)$, .. all share the same sets of roots. (Here

$x$ is allowed to take all possible complex values.) This holds because the exponential function $e^x$ never takes the value 0. For a complete statement of the relevant theorems, the reader should refer to the books by Knopp, Titchmarsh and Conway in Suggested Reading or any equivalent text in complex analysis.

So Euler certainly was lucky; but then Euler had an uncanny intuition about such matters, and he probably knew just when it was safe to reason in this manner, just as Ramanujan did more than a century later. However there is still more to be said. Probably such audacity to some extent provided the impetus for the vast amount of research done in complex analysis in the century following Euler—by Cauchy, Weierstrass, Abel, Gauss, Riemann, : the most powerful analysts known to mathematics. As a result, by the end of the nineteenth century, complex analysis had been placed on an extremely secure foundation. Obviously, this owes in considerable measure to the pioneering work done by Euler.

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## Suggested Reading


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