What *Can* the Answer be?

3. Reciprocal Basis in Two Dimensions and Other Nice Things

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The previous articles of this series were:

1. Elementary Vector Analysis, August 1996.
2. Reciprocal Basis and Dual Vectors, October 1996.

In Part 2 of this series, we saw how dual vectors arose very naturally even in elementary vector analysis. At the end of that article, I stated that dual vectors and the reciprocal basis were very far-reaching concepts. They appear in many different contexts, some of which will be described in the sequel. In this part, we begin with a situation that might appear to be simpler than the case worked out in part 2 but we are in for a surprise!

Reciprocal Basis in Two Dimensions (2D)

Let us recall briefly the essential result found in part 2 of this series: given any three non-coplanar, i.e., *linearly independent* vectors $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ in the familiar three-dimensional or 3D Euclidean space, the *reciprocal basis* comprises three vectors $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ such that $\mathbf{A}\cdot\mathbf{a} = \mathbf{B}\cdot\mathbf{b} = \mathbf{C}\cdot\mathbf{c} = 1$, while $\mathbf{A}\cdot\mathbf{b} = \mathbf{A}\cdot\mathbf{c} = \mathbf{B}\cdot\mathbf{a} = \mathbf{B}\cdot\mathbf{c} = \mathbf{C}\cdot\mathbf{a} = \mathbf{C}\cdot\mathbf{b} = 0$. The three vectors $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ are found to be given by

$$\mathbf{A} = \frac{\mathbf{b}\times\mathbf{c}}{V}, \mathbf{B} = \frac{\mathbf{c}\times\mathbf{a}}{V}, \mathbf{C} = \frac{\mathbf{a}\times\mathbf{b}}{V}. \quad (1)$$

Here V is the scalar triple product $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$; its modulus is the volume of the parallelepiped formed by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ as in *Figure 1*. The expressions in (1) have a pleasing cyclical symmetry.

We now ask: what about the simpler case of *two* dimensions, i.e., a plane? Here we have *two* vectors $\mathbf{a}$ and $\mathbf{b}$ that are not parallel or antiparallel to each other (*Figure 2*). We want to find two other vectors $\mathbf{A}$ and $\mathbf{B}$ in the same plane such that

$$\mathbf{A}\cdot\mathbf{a} = \mathbf{B}\cdot\mathbf{b} = 1 \text{ while } \mathbf{A}\cdot\mathbf{b} = \mathbf{B}\cdot\mathbf{a} = 0 \quad (2)$$

This is easily done if we regard $\mathbf{a}$ and $\mathbf{b}$ as defining the directions of a pair of *oblique* axes in the plane; then $\mathbf{A}$ and $\mathbf{B}$ *must* be linear combinations of the form

$$\mathbf{A} = \alpha_1 \mathbf{a} + \beta_1 \mathbf{b}, \mathbf{B} = \alpha_2 \mathbf{a} + \beta_2 \mathbf{b}. \tag{3}$$

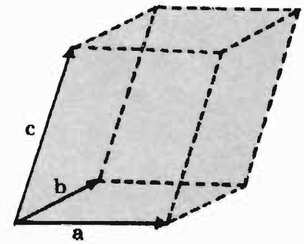


Figure 1

The four constants $\alpha_1, \beta_1, \alpha_2, \beta_2$ can now be found by taking the dot products of $\mathbf{A}$ and $\mathbf{B}$ with $\mathbf{a}$ and $\mathbf{b}$ in turn, and using the values given by (2) for these quantities. This involves solving *four* simultaneous equations, which is a bit tedious, although quite straightforward. Is there a simpler, shorter way? We must not accept the following erroneous argument:

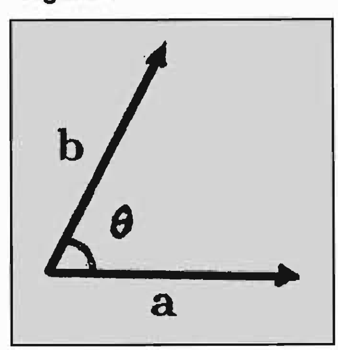
"Since $\mathbf{A}\cdot\mathbf{b}=0$, $\mathbf{A}\perp\mathbf{b}$. Similarly $\mathbf{B}\cdot\mathbf{a}=0$, so that $\mathbf{B}\perp\mathbf{a}$. Hence $\mathbf{A}$ cannot have a part proportional to $\mathbf{b}$, i.e., $\beta_1 = 0$. Similarly $\alpha_2 = 0$ This leaves only the two constants α_1 and β_2 to be determined."

Such an argument is *only* valid if $\mathbf{a}$ and $\mathbf{b}$ are mutually perpendicular! If $\mathbf{a}$ and $\mathbf{b}$ are not perpendicular to each other, then $\mathbf{A} \;\; \mathbf{b} = 0$ does *not* imply that $\mathbf{A}$ is directed along the other axis, namely, $\mathbf{a}$: since $\mathbf{a}$ itself has a perpendicular projection along $\mathbf{b}$, $\mathbf{A}$ cannot be directed exclusively along $\mathbf{a}$. It must have a compensating piece proportional to $\mathbf{b}$ as well, so that its *net* perpendicular projection on $\mathbf{b}$ is zero.

But there *is* a way to find $\mathbf{A}$ and $\mathbf{B}$ by solving just two equations, rather than four. Any arbitrary vector $\mathbf{v}$ in the plane can be expanded in the form

$$\mathbf{v} = c_1 \mathbf{a} + c_2 \mathbf{b}. \tag{4}$$

Figure 2



Now let us recall from part 2 of this series that the objects $\mathbf{a}\,\mathbf{A}$ and $\mathbf{b}\,\mathbf{B}$ also serve as *projection operators* that add up to the unit operator i.e., $\mathbf{aA} + \mathbf{bB} = \mathbf{I}$. This is entirely equivalent to saying that, for any *arbitrary* vector $\mathbf{v}$,

$$\mathbf{v} = \mathbf{I} \;\; \mathbf{v} = \mathbf{a}\,(\mathbf{A} \;\; \mathbf{v}) + \mathbf{b}\,(\mathbf{B} \;\; \mathbf{v}) \tag{5}$$

In other words, $c_1 = \mathbf{A}\cdot\mathbf{v}$ and $c_2 = \mathbf{B}\cdot\mathbf{v}$. Taking the dot products $\mathbf{a}\cdot\mathbf{v}$ and $\mathbf{b}\cdot\mathbf{v}$ in succession in (4), we get two simultaneous equations for c_1 and c_2. Solving these, we get c_1 and c_2. We can now simply *identify* $\mathbf{A}$ and $\mathbf{B}$ from the expressions for c_1 and c_2 exploiting the fact that $c_1 = \mathbf{A}\cdot\mathbf{v}$, $c_2 = \mathbf{B}\cdot\mathbf{v}$. The result is

$$\mathbf{A} = \frac{b^2\mathbf{a} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{b}}{a^2b^2 - (\mathbf{a}\cdot\mathbf{b})^2}, \quad \mathbf{B} = \frac{a^2\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{a}}{a^2b^2 - (\mathbf{a}\cdot\mathbf{b})^2}, \tag{6}$$

where $a^2 = \mathbf{a}\cdot\mathbf{a}$ and $b^2 = \mathbf{b}\cdot\mathbf{b}$

Although these expressions are not too complicated, they are not too simple, either. Nor do they have the elegant cyclically symmetrical form of the expressions in the 3D case, equations (1). This is quite surprising, because we should expect the answer in 2D to be actually simpler than that in 3D! In particular, the denominator $a^2b^2 - (\mathbf{a}\cdot\mathbf{b})^2$ in (6) is of *second* order in $\mathbf{a}$ and $\mathbf{b}$, while the denominator V in (1) is of *first* order in $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. The 2D analogue of the volume V of the parallelepiped formed by ($\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$) in 3D is the *area* — $\mathbf{a}\times\mathbf{b}$ — of the parallelogram formed by ($\mathbf{a}$, $\mathbf{b}$). We should therefore expect this area to appear in the denominator in the formulas for $\mathbf{A}$ and $\mathbf{B}$. The problem, however, is that it is not possible to have a vector or cross-product of two vectors in 2D space, i.e., for vectors living strictly in a plane! More precisely : if (a_1, a_2) and (b_1, b_2) are the components of the 2D vectors $\mathbf{a}$ and $\mathbf{b}$, the cross product '$\mathbf{a}\times\mathbf{b}$' only has *one* component, $a_1b_2 - a_2b_1$, instead of the *two* needed to make a 2D vector. This is the root of the difficulty.

But now we notice something interesting. The *square* of $a_1b_2 - a_2b_1$ is just $a^2b^2 - (\mathbf{a}\cdot\mathbf{b})^2$, remembering that $a^2 = a_1^2 + a_2^2$ and $b^2 = b_1^2 + b_2^2$! And if $\mathbf{A}$ and $\mathbf{B}$ are written out *component-wise*, a factor $a_1b_2 - a_2b_1$ cancels out in each case, and we get:

$$A_1 = \frac{b_2}{a_1b_2 - a_2b_1}, \quad A_2 = \frac{-b_1}{a_1b_2 - a_2b_1} ;$$

$$B_1 = \frac{-a_2}{a_1 b_2 - a_2 b_1}, \quad B_2 = \frac{a_1}{a_1 b_2 - a_2 b_1}. \tag{7}$$

These expressions do show (at last!) a sort of cyclic symmetry: Let us compare them with what happens in 3D, equations (1). In that case we have

$$A_1 = \frac{(b_2 c_3 - b_3 c_2)}{V}, \quad A_2 = \frac{(b_3 c_1 - b_1 c_3)}{V}, \quad A_3 = \frac{(b_1 c_2 - b_2 c_1)}{V} \tag{8}$$

where

$$\begin{aligned} V &= \mathbf{a} \ \ (\mathbf{b} \times \mathbf{c}) \\ &= a_1 (b_2 c_3 - b_3 c_2) + a_2 (b_3 c_1 - b_1 c_3) + a_3 (b_1 c_2 - b_2 c_1), \end{aligned} \tag{9}$$

and similar expressions for the components of **B** and **C**. What is the common feature of the denominators in (7) and (8)? In each case, we have simply the determinant formed by writing out the basis vectors in component form, one after the other, i.e.,

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \ \ \textit{or} \ \ \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \ \ \text{in 2D ;} \tag{10}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \ \ \textit{or} \ \ \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \ \ \text{in 3D} \tag{11}$$

This is the vital clue — the hidden pattern is now sufficiently revealed for us to guess the complete answer in an *arbitrary* number of dimensions! To do that, it is first necessary to introduce an important symbol and a convenient bit of notation.

Levi-Civita's Symbol and Einstein's Convention

We shall use the subscripts $i, j, k, \ldots$ to denote the various components of a vector — e.g., a_i stands for the i th component of the vector **a**. Here the subscript or *index* i can take on values 1 or 2 in 2D; 1, 2 or 3 in 3D; and 1, 2,.., or n in nD.
Now consider the set of $2^2 = 4$ quantities denoted by ε_{ij} in 2D,

and defined as follows: $\varepsilon_{12} = +1$, $\varepsilon_{21} = -1$, $\varepsilon_{11} = \varepsilon_{22} = 0$ Its counterpart in 3D is ε_{ijk}, defined as follows :

$$\varepsilon_{ijk} = \begin{cases} +1, \text{if } ijk \text{ is an } \textit{even} \text{ permutation of } 123 \\ -1, \text{if } ijk \text{ is an } \textit{odd} \text{ permutation of } 123 \\ 0, \text{ in all other cases} \end{cases} \tag{12}$$

(A permutation of 123 is even [or odd] if it is made up of an even [or odd] number of interchanges of two indices at a time.) Thus, of the $3^3 = 27$ quantities ε_{ijk}, we have $\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = +1$, $\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1$, while the remaining 21 quantities are zero. (It is evident that ε_{ijk} is zero whenever at least two of the indices take on the same value, such as ε_{112} or ε_{333}). The generalization to 4,5, ...n dimensions is immediate! In n dimensions, the indices $i, j, k, l, \ldots$ can take on values from 1 to n. Then

$$\varepsilon_{ijkl\ldots} = \begin{cases} +1, \text{if } ijkl\ldots \text{ is an } \textit{even} \text{ permutation} \\ \quad \text{of the natural order } 1234\ldots n \\ -1, \text{if } ijkl\ldots \text{ is an } \textit{odd} \text{ permutation} \\ \quad \text{of the natural order } 1234\ldots n \\ 0, \text{ whenever any two indices are equal.} \end{cases} \tag{13}$$

$\varepsilon_{ijk\ldots}$ is called the *Levi-Civita* (or totally antisymmetric) symbol in n dimensions. We shall see its great utility shortly.

Among other uses, the Levi-Civita symbol helps us write down the volume of the parallelepiped formed by the basis vectors **a**, **b** ...in any number of dimensions, *i.e.*, the value of the determinant formed by the components of the vectors. We see at once

Tullio Levi-Civita (1873-1941), Mathematician

Abraham Pais, in his superb biography of Einstein (*Subtle is the Lord* ...), from which the quotations here are taken, speaks of " a noble line of descendence" in the works of Gauss, Riemann, Christoffel, Ricci and Levi-Civita, one of whose culmination points was Einstein's General Theory of Relativity (GTR). In 1917, Levi-Civita introduced in a mathematically rigorous manner the concept of *parallel transport*, a fundamental notion in tensor calculus and differential geometry. His correspondence with Einstein early in 1915 helped Einstein in his final formulation of GTR later that year - he was "happy to have finally found a professional who took a keen interest in his work" , and in a grateful letter to Levi-Civita, said, ..."It is therefore doubly gladdening to get to know better a man like you".

that in 2D, this is

$$(a_1b_2 - a_2b_1) = \sum_{i=1}^{2} \sum_{j=1}^{2} \varepsilon_{ij} a_i b_j \tag{14}$$

Similarly, in 3D,

$$V = \mathrm{a} \cdot (\mathrm{b} \times \mathrm{c}) = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} a_i b_j c_k \tag{15}$$

The notation is simplified considerably if we adopt a *convention* – namely, to agree that if an index is *repeated* (i.e., appears twice in any expression), *it is automatically summed over all the values it can take*. This summation convention was introduced by Einstein himself in 1916. Besides reducing considerably the 'clutter' in mathematical expressions, it has a great advantage. It gives us a way of making an important consistency check on calculations involving tensors: every index symbol that appears once on the left-hand side of any equation must do so on the right-hand side as well; any index symbol that appears twice in an expression is a 'dummy index', to be summed over all its possible values; and

Einstein's Summation Convention

Mathematical *notation* is generally regarded as a trivial matter. It is often so - and yet, proper notation is so essential for clear understanding ! And there are some striking instances when adopting a good notation has helped vitally in the development of the subject. Newton, when he invented (discovered?!) the differential calculus - which he orginally called 'fluxions' - used $\dot{y}$, $\ddot{y}$, ... to denote successive derivatives. It is easy to see that this notation rapidly leads to problems with higher order derivatives, partial derivatives and so on. In contrast, to quote E T Bell in *Men of Mathematics*, " ... the more progressive Swiss and French, following the lead of Leibniz, and developing his incomparably better way of merely *writing* the calculus, perfected the subject ", and thus made it a "... simple, easily applied implement of research ... ".

Two other instances come to mind in which a happy choice of notation even acts as an automatic check against mistakes: Dirac's bra and ket notation for linear vector spaces, which we introduced in part 2 of this series; and the Einstein summation convention in tensor analysis. (If an index symbol appears twice in an expression, it is to be summed over all its allowed values. If it appears more than twice, there's a mistake somewhere !) Einstein himself appears to have been pleased with his innovation, for he jested to a friend that he had "made a great discovery in mathematics; I have suppressed the summation sign every time that the summation must be made over an index that appears twice ... " !

no index symbol can appear in any expression *more* than twice. In n -dimensions, therefore, the volume of the (hyper-) parallelepiped formed by the basis vectors **a, b, c, d**, ... is simply

$$\varepsilon_{ijkl\,\ldots}\; a_i\, b_j\, c_k\, d_l\, \cdots \tag{16}$$

where each subscript must be summed over from 1 to n

A remark : in the special case of *three* dimensions, and *only* in this case, is the definition of Levi-Civita symbol given in (12) entirely equivalent to saying that $\varepsilon_{ijk} = +1$ if ijk is 123 or a cyclic permutation of 123; $\varepsilon_{ijk} = -1$ if ijk is 132 or a cyclic permutation of 132; and $\varepsilon_{ijk} = 0$ in all other cases. Indeed, this is the definition given in some books. While it is correct, it can be misleading, because it cannot be extended as it stands to any other dimension, including *two* dimensions ($\varepsilon_{12} = +1$, but $\varepsilon_{21} = -1$ although 21 is a cyclic permutation of 12). The correct general definition is that in (13).

Again, *it is only in 3D that the cross-product of two vectors is itself a vector* (I will qualify this remark later on, in Part 4, in the interests of technical accuracy!) It is easy to check that the (k th component of the vector formed by the) cross-product of two vectors **a** and **b** in 3D is just $\varepsilon_{ijk}\, a_i\, b_j$ — this quantity has precisely one 'free' index (namely, k), as required by a vector. On the other hand, in 2D we have $\varepsilon_{ij}\, a_i\, b_j$ which has no free index left at all, and is thus a scalar; while in more than 3D we have $\varepsilon_{ijkl\,\ldots}\, a_i\, b_j$ which has two or more free indices $(k, l, \ldots)$, and hence denotes a 'tensor' of *rank* 2 or more. Since any two (non-collinear) vectors define a plane, a geometrical way of saying all this is as follows: in n -dimensional space, we have n independent directions and ${}^nC_2 = n\,(n-1)/2$ independent planes. Only in 3D are these two numbers equal to each other! This is one of the main reasons why 3D is so special.

We have now set up all the machinery needed to find the reciprocal basis in an arbitrary number of dimensions. This will be our first task in the final part 4 of this series.

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