Think It Over

This section of Resonance *is meant to raise thought-provoking, interesting, or just plain brain-teasing questions every month, and discuss answers a few months later. Readers are welcome to send in suggestions for such questions, solutions to questions already posed, comments on the solutions discussed in the journal, etc. to* Resonance *Indian Academy of Sciences, Bangalore 560 080, with "Think It Over" written on the cover or card to help us sort the correspondence. Due to limitations of space, it may not be possible to use all the material received. However, the coordinators of this section (currently A Sitaram and R Nityananda) will try and select items which best illustrate various ideas and concepts, for inclusion in this section.*

Question raised in the *Think It Over* section of *Resonance*, Vol.1, No.7.

The Odd Ball Problem

The given problem is the special case $n=3$ of the following general result:

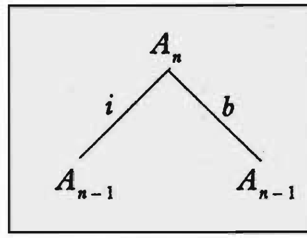
Problem D_n : Given $(3^n - 3)/2$ balls one of which is odd (it is not known whether this odd ball is lighter or heavier than the others), to find the odd ball and determine whether it is heavier or lighter than the rest in n weighings.

We state and solve three different types of problems, A_n, B_n, C_n which are inter-related and the solution of the given problem will follow. This solution is based on the one given by U Leron (see suggested reading). In all the problems a balance with two pans but without weights is provided.

Correct solutions to this problem were also provided by R K Patney, Monish Unni, K C Pradeep and A Uma Shankar.

Problem A_n : Given are 3^n balls all of which except one are of the same weight. The exception is 'known' to be *heavier* (respectively, *lighter*) than the others. We are to find this ball in n weighings.

Solution: This is the easiest of the three problems and the proof is by induction. The case $n = 1$ is very easy and we leave it to the reader to verify. Let us consider the case A_n, $n \geq 2$. Split them into three equal piles of 3^{n-1} balls each. Weigh any two piles against each other. If there results an imbalance, then the heavier pile (respectively, the lighter pile) contains the odd ball and the problem is reduced to A_{n-1}. If there is a balance, then the unused pile has the odd ball and again we are led to A_{n-1}.



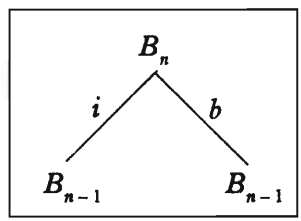
i means imbalance
b means balance

Problem B_n : Given are 3^n balls each coloured red or blue. We 'know' that the odd ball is heavier if it is red and lighter if it is blue. It is required to find the odd ball in n weighings (and the colour would tell us if it is heavier or lighter).

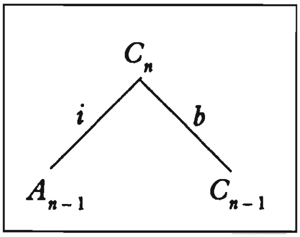
Solution: Again the proof is by induction. If $n=1$, we have two cases:

(i) If all the balls are of the same colour the problem is solved by A_1.

(ii) Two are of one colour and the third of a second colour. Weigh the two balls of the same colour one against the other. An imbalance tells us which pan contains the odd ball from its colour. A balance tells us that the third ball is odd and its colour reveals whether it is heavier or lighter.

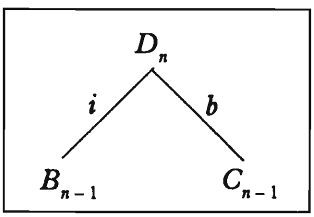


Consider the general case B_n. Since 3^n is odd, one colour occurs an odd number of times and the other an even number of times. We remove one ball having the colour that occurs an odd number of times, so that among the remaining $3^n - 1$ balls both colours occur an even number of times. We further remove pairs of balls of same colour, until we are left with 2 3^{n-1} balls which can be split into two similar piles. We weigh these two piles one against the other. If an imbalance results, we take 'all' red balls from the lowered pan and 'all' blue balls from the raised pan and put these together. They are 3^{n-1} in number and they contain the odd ball. We have reduced the problem to B_{n-1}. If a balance results, again the odd ball is among the remaining 3^{n-1} balls. The problems is again reduced to B_{n-1}.



Problem C_n : Given $(3^n - 1)/2$ balls, one of which is odd (it is not known whether the odd ball is heavy or light), and an adequate supply of good balls (in fact, not more than 3^{n-1} good balls are needed), to find the odd ball and determine whether it is heavier or lighter than the rest in n weighings.

Solution: The case $n=1$ is easy. Simply weigh the given ball against one good ball and we have complete information. Consider the case C_n, $n \geq 2$. We weigh 3^{n-1} of the $(3^n - 1)/2$ balls against 3^{n-1} good balls. If an imbalance results we finish by A_{n-1}. Otherwise, the false ball is among the remaining $(3^n-1)/2 - 3^{n-1} = (3^{n-1} - 1)/2$ balls. So we have reduced the problem to C_{n-1}.



Coming to our original problem, D_n, the least value of n for which D_n is meaningful is $n=2$. In this case we have three balls and it is easy to find out the odd ball in 2 weighings. Now consider the general case D_n, : $n \geq 3$. There are $(3^n - 3)/2$ balls.

Split them into 3 equal piles of $(3^{n-1} - 1)/2$ balls each and weigh any two of them against each other. If an imbalance results colour the heavier pile red and the lighter pile blue. Thus we have $(2/3).\ (1/2).\ (3^n-3) = 3^{n-1}-1$ coloured balls. Throw in any one good ball and we have B_{n-1}. If a balance results, then the odd ball is among the remaining $(3^{n-1} - 1)/2$ balls. Now we finish by C_{n-1}, as we have an adequate supply of good balls, in fact more than the required 3^{n-2} balls.

Remark: When we say colour the balls red or blue, it only means that we have sufficient means of identification. This is needed for a subsequent grouping of the balls.

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Suggested Reading

◆ **U Leron. The false coin problem, a colorful approach. *Int. J. Math. Educ. Sci. Technol.* Vol. 12. No. 3. pp.343-350, 1981.**