Transfinite Numbers

What is Infinity?

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In a series of revolutionary articles written during the last quarter of the nineteenth century, the great German mathematician Georg Cantor removed the age-old mistrust of infinity and created an exceptionally beautiful and useful theory of transfinite numbers. This is an introductory article on this topic.[1]

Introduction

How many points are there on a line? Which is more in number—points on a line or lines in a plane? These are some natural questions that have occurred to us sometime or the other. It is interesting to note the difference between the two questions. Do we have to know how many points and lines there are to answer the second question? Even primitive man could see whether he had more cows than his neighbour without knowing the actual number of cows they had—he would just pair off his cows with his neighbour's. So, to compare the size of two sets, we can start pairing off their elements. If one of them exhausts first we say that that particular set has fewer elements than the other one. Otherwise we say that the two sets have the same number of elements. If we are considering finite collections only, everything seems to be alright. Now imagine yourself walking into a hotel with infinitely many rooms which are all occupied. The receptionist shifts the guest in Room 1 to Room 2, the one in Room 2 to Room 3 and so on. Now Room 1 falls vacant and she gives it to you. As we can pair off the two sets of guests, question arises: are there more guests in the hotel now? Thus we see that the idea of pairing off elements leads to counterintuitive results for infinite sets, e.g., an infinite set can have as many elements as a part of it. Georg Cantor (1845-1918) showed that this is a characteristic difference between finite and infinite sets and created an immensely useful branch of mathematics based on this idea which had a great impact on the whole of mathematics. For example, the question of what is a number (finite or infinite) is almost a philosophical one. However Cantor's work turned it into a precise mathematical concept.
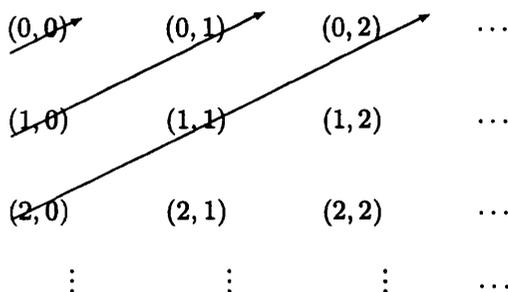
Countable Sets

The simplest sets are those whose elements can be counted as the first, the second, the third etc. with the possibility that the counting may not stop. Such sets are called *countable*. Mathematically, a set A is countable if there is a one-one map from the set N of natural numbers $0, 1, 2, \ldots$, or from $\{0, 1, \cdots n-1\}$ (n being a natural number) onto A. (For $n = 0$ we take the set $\{0, 1, \cdots n-1\}$ to be the empty set.) If a set is not countable we call it *uncountable*. You can easily see that a set is countable if and only if its elements can be enumerated as $a_0, a_1, a_2, \ldots$, may be by repeating some of its elements. It follows that a subset of a countable set is countable.

Examples

•We can enumerate the set $N \times N$ of ordered pairs of natural numbers by the diagonal method as shown in the following diagram

$$(0,0) \qquad (0,1) \qquad (0,2) \qquad \cdots$$

$$(1,0) \qquad (1,1) \qquad (1,2) \qquad \cdots$$

$$(2,0) \qquad (2,1) \qquad (2,2) \qquad \cdots$$

$$\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad \cdots$$

That is, we enumerate the elements of $N \times N$ as $(0,0), (1,0), (0,1),$ $(2,0), (1,1), (0,2), \ldots$. By induction we see that the set of all k-tuples of natural numbers is also countable.

• An argument similar to the one above shows that if $A_0, A_1, A_2, \ldots$ is a sequence of countable sets then their union $A = \bigcup_0^\infty A_n$ is also countable.

• The set Q of all rational numbers is countable. (Exercise: Try to deduce this from the first example.)

The most natural question that arises now is: are there uncountable sets? The answer is yes. Here is an example.

• If $a < b$ are real numbers then the interval $[a, b]$ is uncountable.

Proof. If possible, let $[a, b]$ be countable and let $\{a_0, a_1, \ldots\}$ be an enumeration of $[a, b]$. Define an increasing sequence $\{b_n\}$ and

How many points are there on a line? Which is more in number – points on a line or lines in a plane?

a decreasing sequence $\{c_n\}$ of real numbers in $[a, b]$ inductively as follows: Put $b_0 = a$ and $c_0 = b$. For some $n \in N$, let

$$b_0 < b_1 < \cdots < b_n < c_n < \cdots < c_1 < c_0$$

be defined. Let i_n be the first integer i such that $b_n < a_i < c_n$ and j_n be the first integer j such that $a_{i_n} < a_j < c_n$. Put $b_{n+1} = a_{i_n}$ and $c_{n+1} = a_{j_n}$. Now, the number $x = \sup\{b_n\}$ is in $[a, b]$ but is different from each of a_n. This is a contradiction.

Combining the above observations we easily see that every interval with more than one point contains uncountably many numbers which are not algebraic.[2] Such numbers are called *transcendental*. This simple argument of Cantor generated a lot of interest (as well as scepticism   for not giving an example of a transcendental number but only showing their existence) in set theory at that time. Later Hermite and Lindemann showed respectively that π and e are transcendental.

Equinumerosity—Order of Infinity

Say that sets A and B are *equinumerous* or are of the same *cardinality*, written $A \equiv B$, if there exists a one-one map f from A onto B.

Exercise Show that if I and J are two intervals having more than one point (which may be bounded or unbounded, may or may not include their end points) they have the same cardinality.

In the last section we saw that the idea of equinumerosity, besides being a natural one, is useful too. So let us study it.

Let X and Y be sets. The collection of all subsets of a set X is itself a set called the *power set* of X denoted by $\mathcal{P}(X)$. Similarly the collection of all functions from Y to X forms a set which we denote by X^Y If A is a subset of X then its *characteristic function* is the map $\chi_A : X \longrightarrow \{0, 1\}$ where

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

You can easily verify that $A \longrightarrow \chi_A$ defines a one-one map from $\mathcal{P}(X)$ onto $\{0, 1\}^X$.

So far we have seen only two different 'orders of infinity': those equinumerous either with N or with the set of real numbers $\Re$. Are there any more? Cantor's answer was simple but breathtaking.

[2] An *algebraic number* is a complex number which is a root of a polynomial having rational numbers for coefficients.   For eg: $\sqrt{2}$ is an algebraic number; it satisfies $x^2 - 2 = 0$.

Say that *the cardinality of a set A is less than or equal to the cardinality of set B*, written $A \leq_c B$, if there is a one-one function f from A to B. If $A \leq_c B$ but $A \not\equiv B$ then we say that *the cardinality of A is less than the cardinality of B* and symbolically write $A <_c B$.

Notice that $N <_c \Re$.

Theorem(Cantor) For any set X, $X <_c \mathcal{P}(X)$.

(Note that the power set of the empty set is not empty—it contains the empty set.)

Proof. The map $x \longrightarrow \{x\}$ from X to $\mathcal{P}(X)$ is one-one. Therefore, $X \leq_c \mathcal{P}(X)$. Now take any map $f : X \longrightarrow \mathcal{P}(X)$. We show that f cannot be mapped onto $\mathcal{P}(X)$. To show this consider the set

$$A = \{x \in X \mid x \notin f(x)\}.$$

(A may not have any point, thus it may be an empty set—yet a subset of X).

If $A = f(x_0)$ for some $x_0 \in X$ then note that

$$x_0 \in A \iff x_0 \notin A.$$

This contradiction proves our claim.

Now we see that

$$N <_c \mathcal{P}(N) <_c \mathcal{P}(\mathcal{P}(N)) <_c \cdots.$$

Let T be the union of all the sets N, $\mathcal{P}(N)$, $\mathcal{P}(\mathcal{P}(N))$, $\cdots$. Then T is of cardinality larger than each of the sets described above. We can now similarly proceed with T and get a never ending class of sets of higher cardinalities! Cantor at this point raises a very interesting problem. *Is there an infinite set whose cardinality is different from the cardinalities of each of the sets so obtained?* In particular, is there an uncountable set of real numbers of cardinality less than that of $\Re$? These turned out to be among the most fundamental problems not only in set theory but in the whole of mathematics. Cantor could neither solve these problems nor did he live long enough to see their surprising solution. We shall discuss these later.

The following result is very useful in proving equinumerosity of two sets.

Schröder-Bernstein Theorem. For any two sets X and Y,

$$X \leq_c Y \ \& \ Y \leq_c X \Longrightarrow X \equiv Y.$$

Is there an uncountable set of real numbers of cardinality less than that of $\mathcal{R}$?

Proof. (Dedekind) Fix one-one maps $f : X \longrightarrow Y$

and $g : Y \longrightarrow X$. Consider the map $\mathcal{H} : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ defined by

$$\mathcal{H}(A) = X \setminus g(Y \setminus f(A)), \quad A \subset X.$$

Then

    i) $A \subset B \subset X \Longrightarrow \mathcal{H}(A) \subset \mathcal{H}(B)$, and
    ii) $\mathcal{H}(\bigcup_n A_n) = \bigcup_n \mathcal{H}(A_n)$.

Now, let

    $A_0 = \emptyset$, and
    $A_{n+1} = \mathcal{H}(A_n)$, $n = 0, 1, 2, \ldots$.

Let $E = \bigcup_n A_n$. Then, $\mathcal{H}(E) = E$. Define $h : X \to Y$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in E \\ g^{-1}(x) & \text{otherwise.} \end{cases}$$

The map $h : X \to Y$ is easily seen to be one-one and onto.

Here are some applications of Schröder-Bernstein theorem.

• The function $A \longrightarrow \sum_{n \in A} \frac{2}{3^{n+1}}$ from $\mathcal{P}(N)$ to $\Re$ is one-one. Therefore, $\mathcal{P}(N) \leq_c \Re$. On the other hand the map $x \longrightarrow \{r \in Q | r < x\}$ from $\Re$ to $\mathcal{P}(Q)$ is one-one and so, $\Re \leq_c \mathcal{P}(Q)$. But $Q \equiv N$. Therefore, $\Re \leq_c \mathcal{P}(N)$. By Schröder-Bernstein theorem, $\Re \equiv \mathcal{P}(N)$ $(\equiv \{0,1\}^N)$.

• Fix a one-one map $x \longrightarrow (x_0, x_1, x_2, \ldots)$ from $\Re$ onto $\{0,1\}^N$, the set of sequences of 0's and 1's. Then the function $(x, y) \longrightarrow (x_0, y_0, x_1, y_1, \ldots)$ of the pairs of real variables to $\{0,1\}^N$ is one-one and onto. So, $\Re^2 \equiv \Re$. Thus there are precisely as many points on the X-axis as in the entire plane! By induction on positive integers k you can now show that $\Re^k$ and $\Re$ are equinumerous. Similarly, using $N \times N \equiv N$, you can show that $\Re$ and $\Re^N$, the set of sequences of real numbers are equinumerous.

• Show that the sets of points on a line and lines in a plane are equinumerous.

The Axiom of Choice

Are the sizes of any two sets necessarily comparable? That is, if X, Y are sets, is it true that at least one of the relations $X \leq_c Y$ or $Y \leq_c X$ holds? To answer this question, we need a hypothesis on sets known as the axiom of choice.

The Axiom of Choice (AC): If $\{A_i\}_{i \in I}$ is a family of non-empty sets then there is a function $f : I \longrightarrow \bigcup_i A_i$ such that $f(i) \in A_i$ for every $i \in I$.

Such a function f is called a choice function. Note that if I is finite then by induction on the number of elements in I we can show that a choice function exists. If I is infinite then we do not know how to prove the existence of such a map. The problem can be explained by the following example of Russell. Let $A_0, A_1, A_2, \ldots$ be a sequence of pairs of shoes. Let $f(n)$ be the left shoe in the n-th pair A_n and so the choice function in this case certainly exists. Instead, let $A_0, A_1, A_2, \ldots$ be a sequence of pairs of socks. Now we are unable to give a rule to define a choice function for the sequence of socks $A_0, A_1, A_2, \ldots$! AC asserts the existence of a function without giving any rule or any construction for defining such a function. Because of its non-constructive nature, AC met with serious criticism. For an excellent account of AC see Gregory Moore's book mentioned in Suggested Readings. We only remark that AC is indispensable not only for the theory of cardinal numbers but for the whole of mathematics. Note that while proving that the union of a sequence of countable sets $A_0, A_1, \ldots$ is countable we have used AC. For each n, we chose an enumeration of A_n. But there are infinitely many such enumerations and we did not specify any rule to choose them. The next result shows that every infinite set has a proper equinumerous subset. In its proof we use AC.

Theorem If X is infinite and $A \subset X$ finite then $X \setminus A$ and X have the same cardinality.

Proof. By AC, fix a choice function $f : \mathcal{P}(X) \setminus \{\emptyset\} \longrightarrow X$ such that $f(E) \in E$ for every non-empty subset E of X. Let $A = \{a_0, a_1, \ldots, a_n\}$. Inductively define $a_{n+1}, a_{n+2}, \ldots$ such that

$$a_{n+k+1} = f(X \setminus \{a_0, a_1, \ldots, a_{n+k}\}).$$

Now define $h : X \longrightarrow X \setminus A$ by

$$h(x) = \begin{cases} a_{n+k+1} & \text{if } x = a_k \\ x & \text{otherwise.} \end{cases}$$

Clearly, $h : X \longrightarrow X \setminus A$ is one-one and onto.

There are many equivalent forms of AC. We shall use Zorn's lemma—one of its equivalent forms—to prove further results. We explain Zorn's lemma now.

A *partial order* on a set P is a binary relation R such that for $x, y, z \in P$,

i) xRx (reflexive),

ii) xRy and yRz implies xRz (transitive), and

iii) xRy and yRx implies $x = y$ (anti-symmetric).

A set P with a partial order is called a *partially ordered set* or simply a *poset*. A *linear order* on a set X is a partial order R on X such that any two elements of X are comparable, i.e., for every $x, y \in X$, at least one of xRy or yRx holds. If X is a set with more than one element then the inclusion relation $\subset$ on $\mathcal{P}(X)$ is a partial order which is not a linear order. Another example of a partial order which is not a linear order is obtained as follows. Let X and Y be sets with Y having more than one element. Let $\mathcal{P}$ be the set of all partial functions from X to Y (functions with domain a subset of X and range contained in Y). Order $\mathcal{P}$ by 'extension' (if $f, g \in \mathcal{P}$ then g extends f if the domain of f is contained in the domain of g and $f(x) = g(x)$ for every x in the domain of f).

Fix a poset (P, R). A *chain* in P is a subset C of P such that R restricted to C is a linear order, i.e., for any two elements x, y of C either xRy or yRx must be satisfied. If $A \subset P$ then an *upper bound* for A is an $x \in P$ such that yRx for all $y \in A$. An $x \in P$ is called a *maximal element* of P if for no $y \in P$ different from x, xRy holds. In the second example above, any partial function f with domain X or range Y is a maximal element. So, there may be many maximal elements in a poset which is not linearly ordered.

Zorn's Lemma: If P is a non-empty partially ordered set such that every chain in P has an upper bound in P then P has a maximal element.

Some Consequences of the Axiom of Choice

Theorem For any two sets X and Y at least one of the relations $X \leq_c Y$ or $Y \leq_c X$ holds.

Proof. Without loss of generality we can assume that both X and Y are non-empty. We need to show that either there exists a one-one map $f : X \longrightarrow Y$ or there exists a one-one map $g : Y \longrightarrow X$. Towards this let P be the set of all one-one partial functions from X to Y. The set P is clearly non-empty. Order it by 'extension'. Let $C = \{f_i : i \in I\}$ be a chain in P. Then C is a consistent family of one-one partial functions. Let $D = \bigcup_{i \in I} \operatorname{domain}(f_i)$. Define $f : D \longrightarrow Y$ by

$$f(x) = f_i(x) \quad \text{if} \quad x \in \operatorname{domain}(f_i).$$

Since f_i's are consistent and one-one, f is well defined and one-one. Thus, f is an upper bound of C. By Zorn's lemma, P has a maximal element, say f_0. We must have domain(f_0) = X or range$(f_0) = Y$. Otherwise, choose $a \in X \backslash$ domain(f_0) and $b \in Y \backslash$ range(f_0). We get a one-one extension of f_0 to domain$(f_0) \bigcup \{a\}$ by assigning b to a. This contradicts the maximality of f_0. If domain$(f_0) = X$ then $X \leq_c Y$ and if range$(f_0) = Y$ then $Y \leq_c X$.

The next result shows that a hotel with infinitely many rooms even when full can accommodate as many new guests as are already there!

Theorem For every infinite set X, $X \times \{0,1\} \equiv X$.

Proof. Let

$$P = \{(A, f) : A \subset X \quad \text{and} \quad f : A \times \{0,1\} \longrightarrow A \quad \text{a bijection}\}.$$

Since X is infinite it contains a countably infinite set, say D. Since, $D \times \{0,1\} \equiv D$, P is non-empty. Partially order P by 'extension'. As before, by Zorn's lemma, get a maximal element (A, f) of P. The result will follow if we show that $A \equiv X$. Since X is infinite, by the theorem proved in the last section, it will be sufficient to show that $X \backslash A$ is finite. Suppose not. Then by AC there is a $B \subset X \backslash A$ such that $B \equiv N$. So, there is a one-one map g from $B \times \{0,1\}$ onto B. Combining f and g we get a bijection $h : (A \bigcup B) \times \{0,1\} \longrightarrow A \bigcup B$ which extends f. This contradicts the maximality of (A, f). Hence, $X \backslash A$ is finite. Therefore, $A \equiv X$. The proof is complete.

Corollary Every infinite set can be written as the union of k-many pairwise disjoint equinumerous sets where k is any positive integer.

Theorem If X is an infinite set then $X \times X \equiv X$.

Proof. Let

$$P = \{(A, f) : A \subset X \quad \text{and} \quad f : A \times A \longrightarrow A \quad \text{a bijection}\}.$$

Partially order P by 'extension'. By Zorn's lemma take a maximal element (A, f) of P. Note that A must be infinite. To complete the proof, we shall show that $A \equiv X$. Suppose not. Then $A <_c X$.

We first show that $X \backslash A \equiv X$. If possible, suppose $X \backslash A <_c X$. Either $A \leq_c X \backslash A$ or $X \backslash A \leq_c A$. Assume first $X \backslash A \leq_c A$. Take two disjoint sets A_1, A_2 of the same cardinality as A. Now,

A hotel with infinitely many rooms even when full can accommodate as many new guests as already there!

$$X = A \bigcup (X \setminus A) \leq_c A_1 \bigcup A_2 \equiv A <_c X.$$

This is a contradiction. Similarly using the other inequality we arrive at the same contradiction. Thus $X \setminus A \equiv X$. Now get $B \subset X \setminus A$ such that $B \equiv A$. Using the last result, write B as the union of three disjoint sets say B_1, B_2 and B_3 each of the same cardinality as A. Since there is a bijection from $A \times A$ to A there exist bijections $f_1 : B \times A \longrightarrow B_1$, $f_2 : B \times B \longrightarrow B_2$ and $f_3 : A \times B \longrightarrow B_3$. Let $C = A \bigcup B$. Combining these four bijections we get a bijection $g : C \times C \longrightarrow C$ which extends f. This contradicts the maximality of f.

Exercise Show that an infinite set X is equinumerous to the set of all its finite subsets.

The Continuum Hypothesis (CH):

We now get back to the two famous problems of Cantor mentioned earlier. He conjectured the following. (We are assuming AC.)

The Continuum Hypothesis (CH) Every uncountable subset of $\Re$ is of the same cardinality as $\Re$.

The Generalised Continuum Hypothesis (GCH) If X is an infinite set then there is no set A such that $X <_c A <_c \mathcal{P}(X)$.

CH says that $\Re$ is the least numerous uncountable set. If GCH were true then every infinite set is equinumerous to one of the sets obtained by iterating the power set operation on N. Are these statements true? These turned out to be very hard problems and their solutions were startling. In 1938 Kurt Gödel obtained deep results on models of set theory and showed that based on certain axioms for sets (the so called *Zermelo-Fraenkel axioms*), CH and GCH cannot be disproved by producing a 'model' of set theory satisfying the Zermelo-Fraenkel axioms where CH and GCH are satisfied. This was the first time metamathematics entered in a non-trivial way to answer a problem in mathematics. Gödel's result does not say that CH and GCH can be proved. In 1963 Paul Cohen developed a very powerful technique, known as **forcing**, to build models of set theory and

showed that CH and GCH cannot be proved either. A similar and equally famous result was proved in the nineteenth century. It was shown that the axiom of parallels—through a point outside a given line passes a unique line not intersecting the given line—can neither be proved nor can be disproved. (See K Paranjape's series of articles on Geometry in this journal for details.) Gödel and Cohen's results are generally considered to be among the greatest in twentieth century mathematics. Cohen's method is particularly useful in proving such consistency results. A further discussion on this is beyond the scope of this article.

Arithmetic of Cardinal Numbers

For sets X, Y and Z we can easily check the following.
  i) $X \equiv X$,

  ii) if $X \equiv Y$ then $Y \equiv X$, and

  iii) if $X \equiv Y$ and $Y \equiv Z$ then $X \equiv Z$.

So, to each set X we can assign a symbol, say $|X|$, called its cardinal number such that

$$X \equiv Y \iff |X| \text{ and } |Y| \text{ are the same.}$$

The symbols $|X|$ are called *cardinal numbers* or simply *cardinals*. (Cardinal numbers are denoted by Greek letters κ, λ, μ with or without suffixes.) The cardinal number of a finite set with n elements is denoted by n itself. We put $|N| = \aleph_0$ and $|\Re| = c$.

As in the case of natural numbers, we can add, multiply and compare cardinal numbers. Towards this, let λ and μ be two cardinal numbers and X and Y be two sets such that $|X| = \lambda$ and $|Y| = \mu$.

Definition

  a) $\lambda + \mu = |(X \times \{0\}) \cup (Y \times \{1\})|$

  b) $\lambda . \mu = |X \times Y|$

  c) $\lambda^\mu = |X^Y|$

  d) $\lambda \leq \mu$ if $X \leq_c Y$

  e) $\lambda < \mu$ if $X <_c Y$

The above definitions are easily seen to be independent of the choices of X and Y Further, these extend the corresponding notions for natural numbers. We can extend the notion to that of sum and product for infinite collection of cardinals too. Let $\{\lambda_i\}$ be a set of cardinal numbers. Fix a family $\{X_i\}$ of sets of cardinality λ_i. We define $\Pi_i \lambda_i = |\times_i X_i|$. To define $\sum_i \lambda_i$ we take X_i's to be pairwise disjoint and put $\sum_i \lambda_i = |\bigcup_i X_i|$. In terms of these notations note that $\aleph_0.\aleph_0 = \aleph_0$, $2^{\aleph_0} = c$, $\aleph_0 < c$, $c^{\aleph_0} = c$ etc.

Cardinal arithmetic is very similar to the arithmetic of natural numbers and follows more or less the same laws. Whatever we have proved about equinumerosity of sets, i.e., the results concerning union, product, $\leq_c$ etc., translate into corresponding results about cardinal numbers. Thus we obtain a very beautiful and useful cardinal arithmetic, remarkably similar to the arithmetic of natural numbers. For instance, the Schröder-Bernstein theorem translates as follows: $\lambda \leq \mu$ & $\mu \leq \lambda \Longrightarrow \lambda = \mu$; the result on comparability of cardinals becomes: for.

cardinals λ and μ at least one of $\lambda \leq \mu$ or $\mu \leq \lambda$ holds. We get some notable differences for infinite cardinals: if λ is infinite then $\lambda = \lambda + \lambda = \lambda.\lambda$.

Here is how we do cardinal arithmetic.

$$
\begin{aligned}
2^c &\leq c^c & \text{(since } 2 \leq c\text{)} \\
&= (2^{\aleph_0})^c & \text{(since } c = 2^{\aleph_0}\text{)} \\
&= 2^{\aleph_0.c} & \text{(since, for non-empty sets} \\
& & X, Y, Z, (X^Y)^Z \equiv X^{Y \times Z}) \\
&\leq 2^{c.c} & \text{(since } \aleph_0 < c\text{)} \\
&= 2^c & \text{(since } c.c = c\text{).}
\end{aligned}
$$

So, by the Schröder-Bernstein theorem $2^c = c^c$. From this we conclude that there are as many 0-1 valued maps of real numbers as real valued ones!

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Suggested Reading

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