Classroom

In this section of Resonance, *we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.*

Impressions of the 37th IMO - '96

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For those of us who have just entered the university, the 37th IMO was the first and last such Olympiad open to competition. It was, therefore, a once-in-a-lifetime experience. However, what made it even more memorable was the chance it gave us to understand the challenges of mathematics, on the one hand, and the uncomplicated friendliness of mathematicians, on the other. The Olympiad was, in general well organized and created the right atmosphere in which we could enjoy the thrills and tensions of competition during the crucial problem sessions and, thereafter, the pleasure of relaxation amid the friendly chatter of fellow participants.

To some extent, we had been prepared for this pattern of encounter (with tough problems and gentle people) at a pre-IMO camp in Pune for one week before the IMO. That was a week crammed with problem sessions and practice tests of $3\frac{1}{2}$ hours duration which got us into shape for the great event. We also relaxed with computer games in the evenings, and had an enjoyable tour of Pune before leaving for Mumbai by bus on the afternoon of 8th July. The faculty at the Pune camp and professors

Resonance, IMO questions appeared in Vol.1, No. 8 (Aug. '96), pp 90-91 at the IISc, Bangalore, with whom I have interacted through the Regional and National Math Olympiads have been unfailingly kind, generous and helpful. I could not have achieved so much without their guiding influence.

At the Olympiad in Mumbai, it was fun talking to participants from other countries. An interesting part of this interaction was getting books on the national Math Olympiads of different countries. I am happy to have got a book on Slovenian Math. Olympiads. We learnt that most of the participants (notably those from China and Iran) will be taking up Math studies at university. They also said that their governments encouraged them to pursue their specific interests in Mathematics.

If it was interesting meeting Math-buffs from all over the world, the academic side of the Olympiad was no less fascinating. The nature of the problems we were expected to solve sets this particular Olympiad apart from previous such events. The quality of the problems at the 37th IMO was vastly superior to any that had appeared previously. Some of the problems were very interesting. Considering the performances of various teams, we can also state that it was the toughest ever IMO. Generally, an IMO would have four easy problems, one moderately tough and one very tough problem. But this IMO had no such thing as an easy problem. Consequently it was all the more challenging as the following discussion of theproblems [2] and their possible solutions will confirm.

[2] for the text of the problem please see *Resonance*, Vol.1, No.8. (Aug. 1996), pp.90-91.

Problem 1

Part A: To show that the task cannot be accomplished if $2|r$ or $3|r$, many of us felt it ideal to look for 'parity' or rather some invariant property of each move (mod 2) or (mod 3), depending on the case to be taken. One can quickly identify such property in the given situation:

If $2|r$ & $r = a^2 + b^2$ then $|a \pm b|$ is even.

$$\text{If } 3|r \ \& \ r = a^2 + b^2 \text{ then } 3|a \ \& \ 3|b.$$

Once these facts are observed, then $|x - y|$ (where (x, y) is the displacement from B) can be seen to be invariant (mod 2) (mod 3). That takes care of the first part.

Part B: To show that task can be done if $r=73$, we write $73 = 8^2 + 3^2$ and observe that each move changes the position parallel to **AB** by ± 8 and perpendicular to AB by ± 3 *or* vice-versa. By trial and error we can get a sequence of moves.

Part C: I was not able to solve this part completely in the Olympiad. I wrote $97 = 9^2 + 4^2$ and tried for a constructionas in Part (B) but noted that the coin would have to be moved outside the board for each of the constructions. I came to the conclusion that the task cannot be accomplished for $r = 97$.

Problem 2

This was a geometry problem where I was not able to progress much except for one correct step. One of our team members, Rishi Raj, (now in Std X, Ranchi) got a perfect solution using Bretschneider's theorem. Kaustubh Deshmukh also got a perfect solution to this problem.

Problem 3

This was indeed very interesting! Generally, in functional equations problems, the given functional equation will have one or two solutions, one of them sometimes being trivial. But the given functional equation $f(m + f(n)) = f(f(m)) + f(n)$ had not one or two but infinitely many solutions! Even more surprising, the solutions can be constructed with a great degree of arbitrariness!

The basic line of attack is to show $f(0)=0$ and then $f(f(m)) = f(m)$ for $m \in \mathbf{N} \cup \{0\}$. This leads us to consider the fixed points of f, i.e., points j such that $f(j)=j$, and show that all members of the range are multiples of the minimum positive fixed point j_0, if there is one. If 0 is the only fixed point then f is the zero function, i.e., $f \equiv 0$. If 1 is the minimum fixed point then f is the identity function.

Thus choose an arbitrary positive integer j_0 and let $f\ (j_0)=j_0$. For $1 \le r \le j_0$ define $f(r) = k_r\ j_0$, where k_r are arbitrary non-negative integers and extend f to all of $\mathbf{N}$ using the functional equation. These arbitrarily defined f give all solutions. While writing solution in the exam, I took 3 cases, $f(1)=0$, $f(1)=1$, and, $f(1)>1$. I hit upon the correct solution in the first two cases but made a minor error in the last case when I wrote that $f(1)$ is the smallest fixed point (which is not necessarily true).

Problem 4:

This problem can be attacked by solving simultaneously for a and b in terms of k^2 & l^2 and using the fact $a,b \in \mathbf{Z}^+$ to get $13 \mid k^2 + 5\,l^2$ and $37 \mid l^2 \pm 6k^2$. Then we read the orbits of squares (mod 13) and (mod 37) to come to the conclusion that $13^2 . 37^2 \mid k^2$ and l^2 and get the answer as 481^2. (Incidentally, the no: 37 is the number of the IMO.)

Problem 5:

I was able to deal with some cases in this problem, but did not get a complete solution. This was the toughest problem in the whole IMO; only 6 of the 424 students solved it completely.

Problem 6:

This is also a very interesting problem. To begin with note that we can assume without any loss of generality that gcd $(p, q)=1$. I solved the problem by proving a stronger statement, namely, that if we have sequence of $n > p + q$ terms $\langle y_i \rangle$ such that

$\sum_{i=1}^{n} y_i = 0$ and $y_r \in \{p, -q\}$ for $1 \le r \le n$ then there exists j such that $\sum_{i=j+1}^{j+p+q} y_i = 0$ This is done by a pigeonhole type argument. We can show that the number of j such that $y_j = -q$ is divisible by p and is $\ge 2p$; say, $y_{a_0}, y_{a_1}, y_{a_2}, \ldots, y_{a_{rp}}, r \ge 2$ are the y_i's which are equal to $-q$ (y_{a_0} is dummy). Define s_j to be the number of y_i's between j and $j+p$ such that $y_i = p$. Note that if $s_j = q$ for some j then we are through. Then I separately considered the three cases: (i) $s_j > q$ for all j; (ii) $s_j < q$, $0 \le j \le (r-1)p$; (iii) there is a j with $s_j > q$ and another j with $s_j < q$.

The case (i) is not possible (pigeonhole principle); in case (ii) the required sequence can be found essentially by using pigeonhole principle. In case (iii) the required sequence can be found by considering a j with $s_j < q$, $s_{j+1} > q$ or a j with $s_j > q$, $s_{j+1} < q$.

I made a mistake in proving (ii) by wrongly writing $rq - r(q-1) = q$ and this led to a wrong argument for proof of(ii).

It is also noteworthy that although most of the problems in this IMO were tough, they all had beautiful elementary solutions. Till now I know of no solution using higher maths. I felt that this was an ideal IMO paper though the members of many other teams complained that it was too tough. I thoroughly enjoyed myself in the exam hall where the availability of tea, cakes and coke allowed some refreshing moments to balance the stress of mental effort. The challenge of tackling tough problems was every bit as enjoyable as making new friends. I feel that by setting this paper our Indian *Problem Shortlisting Committee* has shown the world what an IMO paper really should be like and what standards should be reached in the future.

It is interesting to note that the average age of the gold medalists in an IMO is less than that of the silver medalists which in turn is less than that of the bronze medalists. That is, younger the student greater the chance of her/his getting a better medal!