Nakula's Extraordinary Talent

Mathematics From the Mahabharatha!

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An attempt is made here to provide a mathematical model which can explain an intriguing talent of Nakula. On the way, some simple and interesting properties of the real number system and related material are discussed.

Introduction

One of the exciting characters in the epic Mahabharatha, one that has fascinated me most since my childhood, is Nakula (the fourth of the Pandava brothers). Especially intriguing was his *ability to travel in rain riding on a horse, without getting wet*. Generally speaking, not everyone is aware of this great skill of Nakula as against the common knowledge that he could tame and train any kind of horse to any degree of perfection. None of the host of Sanskrit or Telugu pundits/scholars I asked, over the decades, could offer a plausible explanation. In fact, most of them looked surprised and wondered why I was seeking justification instead of leaving it aside as the product of a fertile imagination!

In this article, I would like to present a simple mathematical model as a possible motivation based on which Veda Vyasa might have attributed this extraordinary talent to Nakula. This will of course imply that Veda Vyasa was a mathematician! Surely. Why not? Even an Applied Topologist for that matter!

The Model

Assume that rain is falling on the floor sharply at points both of whose coordinates are rational numbers (with respect to a chosen set of coordinate axes). This means that it is raining rather *densely*. The simple theorem in topology about the *arcwise* *connectedness* of the complement of a *countable subset* of the Euclidean space $\mathbf{R}^n$ $(n \geq 2)$ now implies that Nakula (shrunk to a point) can travel continuously from one dry point P to another dry point Q without getting wet. In fact, he can even travel smoothly, for example, along a suitable circular arc joining P and Q. There are *uncountably* many such circular arcs joining P and Q, implying that the choice is indeed *abundant*.

Real Number System is a set which can be geometrically thought of as the points on a straight line.

For those who can understand the words used above, there is no need to say more! Nevertheless, let me take the opportunity to explain the concepts used in the model for the benefit of others.

Real Numbers

We assume that the reader is familiar with the '*Real Number System*'. It is a set, denoted by $\mathbf{R}$, which can be geometrically thought of as the points on a straight line. For this reason, $\mathbf{R}$ is also referred to as the '*Real Line*'. It is a disjoint union of '*rational*' and '*irrational*' numbers. Rational numbers are the fractions m/n of integers m and n with $n \geq 0$. The set of all rational numbers is denoted by $\mathbf{Q}$. It is obvious that the sums and products of rational numbers are again rational. This is not true of the irrational numbers. The irrational numbers are not easy to describe or visualise. Some examples of irrational numbers (but not so obvious) are $\sqrt{2}$, $\sqrt{1.1}$, $\sqrt[3]{2}$ and so on.

Decimal Expansions

Perhaps, the simplest way to distinguish between the rational and irrational numbers is by means of the *decimal expansion* of real numbers, i.e., we assume that every real number x (rational or not) admits a decimal expansion in the form $x = x_0 . x_1 x_2 \ldots x_m \ldots$ where x_0 is the *integral part* (i.e., the largest integer not greater than x, usually denoted by $[x]$) and x_1, x_2, ... are the decimal digits of x all of which are integers lying between 0 and 9. This is easy to prove using simple properties of the convergence of

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series of non-negative real numbers. Now we have the following.

Theorem: A real number $x = x_0 . x_1 x_2 \ldots x_m \ldots$ is rational if and only if its decimal digits recur in a block of some size $n \geq 1$ beyond some stage m, i.e., the block of n digits $x_{m+1} x_{m+2} \ldots x_{m+n}$ recurs indefinitely after the m^{th} decimal place. Indicating the recurring block under the bar, this is usually written as

$$x = x_0 . x_1 x_2 \ldots x_m \overline{x_{m+1} x_{m+2} \ldots x_{m+n}}$$

In other words, a rational number is completely known in principle if its decimals are known for a sufficiently large number of places. The stage will of course depend on the particular number. For instance, $0 = 0.\overline{0}$, $1/3 = 0.\overline{3}$, $1/6 = 0.1\overline{6}$, $22/7 = 3.\overline{142857}$, $1/9 = 0.\overline{1}$, , $1 = 1.\overline{0} = 0.\overline{9}$, etc., are all rational numbers whereas the numbers like 0.010110111011110....., 0.1011001110001111 0000 ... , etc., are irrational.

What are the decimal expansions of $\sqrt{2}$, $\sqrt{1.1}$, $\sqrt[3]{2}$, π etc.? Despite the existence of the decimal expansions for real numbers, it may not be possible to give a rule for finding the decimal expansion of a given irrational number, be it as simple in appearance as $\sqrt{2}$, or as mysterious as π. Several mathematicians, including our legendary Srinivasa Ramanujan, wrote down the decimal expansion of π to a very large number of places.

Density of Rational Numbers

It is fairly easy to see that between any two distinct real numbers there are infinitely many rational as well as irrational numbers. In particular, given any real number x and a positive real number ε, there are infinitely many rational as well irrational numbers between $x - \varepsilon$ and $x + \varepsilon$ (however small ε may be). This property means in mathematical terms that the subset of rational (or irrational) numbers is *dense* in the set of real numbers **R**.

Countable and Uncountable Sets

A set X is said to be *countable* if X is either finite or all of its elements can be *labelled* (or *enumerated*) as $x_1, x_2, \ldots, x_n, \ldots, \ldots,$ labelled one for each of the natural numbers $1, 2, \ldots, n, \ldots$. A set X is countable if its elements are counted in some order as we count the natural numbers. A set which is *not* countable is said to be *uncountable*. Uncountable sets are obviously infinite. Intuitively speaking, among infinite sets, countability or uncountability can be used as a tool to compare the infinitudes.

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Countability of Rational Numbers

The sets like the natural numbers $\mathbf{N} = \{1, 2, \ldots\}$, the integers $\mathbf{Z} = \{0, \pm 1, \pm 2, \ldots\}$, the rational numbers

$\left\{\left\{\frac{m}{n} \mid m, n \in \mathbf{Z}, n \neq 0\right\}\right.$ etc., are all infinite of the same order,

namely, countable. An enumeration for $\mathbf{Z}$ may be taken as for instance $\{0, 1, -1, 2, -2, 3, -3, \ldots, n, -n, \ldots\}$. An enumeration for $\mathbf{Q}$ can be taken as $\{0, x_1, -x_1, x_2, -x_2, \ldots, x_n, -x_n, \ldots\}$ if the subset of all positive rational numbers is enumerated as $\{x_1, x_2, \ldots, x_n, \ldots\}$ This is done diagramatically as follows:

$$
\begin{array}{cccccc}
1, & 2, \longrightarrow & 3, & \cdots & n, & \cdots \\
\downarrow & & & & & \\
\frac{1}{2}, & \frac{2}{2}, & \frac{3}{2}, & \cdots & \frac{n}{2}, & \cdots \\
\frac{1}{3}, & \frac{2}{3}, & \frac{3}{3}, & \cdots & \frac{n}{3}, & \\
\downarrow & & & & & \\
\vdots & & & \vdots & \vdots & \cdots \\
\frac{1}{m}, & \frac{2}{m}, & \frac{3}{m}, & \cdots & \frac{n}{m}, & \cdots \\
\vdots & & & \vdots & \vdots &
\end{array}
$$

Finite Cartesian product of countable sets is countable but infinite Cartesian product of even finite sets need not be countable.

In this arrangement, every fraction $\frac{n}{m}$ appears at least once. Allowing repetitions (for example, 1 can appear again and again as 2/2, 3/3, and so on), we can enumerate them as

$$1;\frac{1}{2},2;3,\frac{2}{3},\frac{1}{3};\frac{1}{4},\frac{2}{3},\frac{3}{2},4;\cdots,n-1;n,\frac{n-1}{2},\frac{n-2}{3},\cdots,\frac{2}{n-1},\frac{1}{n};\frac{1}{n+1},\cdots$$

If $X=\{z_1,z_2,\ldots,z_n,\ldots\}$ is countable, so is the set $X\times X$ of pairs of elements (z_m,z_n) formed from the set X. An enumeration for $X\times X$ may be taken as

$$(z_1,z_1);\ (z_1,z_2),(z_2,z_1);\ (z_1,z_3),(z_2,z_2),(z_3,z_1);\ldots$$
$$\ldots;\ (z_1,z_n),(z_2,z_{n-1}),\ldots,(z_{n-1},z_2)\,(z_n,z_1);\ldots$$

This arrangement is almost like what is done above for positive rational numbers. In particular, the set $\mathbf{Q}\times\mathbf{Q}$ of pairs of rational numbers is countable since $\mathbf{Q}$ is countable.

Some basic facts that can be proved on the same lines as above are the following.

(1) A subset of a countable set is countable.
(2) Countable union of countable sets is countable.
(3) Finite Cartesian product of countable sets is countable but infinite Cartesian product of even finite sets need not be countable, as will be made clear below.

Uncountability of Real Numbers

Now for an example of an *uncountable* set, consider the set T of all real numbers between 0 and 0.2 consisting of those whose decimal expansion contains only the digits 0 and 1, i.e.,

$$T=\{0.a_1\,a_2\,a_3\ldots a_n\ldots \mid a_i=0 \text{ or } 1 \text{ for } i=1,2,\ldots\}$$

For instance, T contains numbers like $0.\overline{0}$, $0.1\overline{0}$, $0.101\overline{0}$, 0.010011000111000011111, $0.\overline{1}$ etc.

What is obvious is that T is infinite but what is not at all clear is the fact that T is indeed uncountable! To see this, let T be enumerated if possible, as $T = \{d_1, d_2, d_3, \ldots, d_n, \ldots\}$. Let

$$\begin{array}{rcl} d_1 & = & 0.d_{11}d_{12}\cdots d_{1n}\cdot \\ d_2 & = & 0.d_{21}d_{22}\cdots d_{2n}\cdots \\ \vdots & & \quad\vdots \qquad \vdots \\ d_m & = & 0.d_{m1}d_{m2}\cdots d_{mn}\cdots \\ \vdots & & \quad\vdots \qquad \vdots \end{array}$$

where each of the decimal digits $d_{11}, d_{12}, \ldots; d_{21}, d_{22}, \ldots$ is either 0 or 1. Now look at the number $x = 0.x_1 x_2 \ldots x_r \ldots$ where

$$x_1 = \begin{cases} 0 & \text{if } d_{11} = 1 \\ 1 & \text{if } d_{11} = 0 \end{cases} \ldots x_2 = \begin{cases} 0 & \text{if } d_{22} = 1 \\ 1 & \text{if } d_{22} = 0 \end{cases} x_r = \begin{cases} 0 & \text{if } d_{rr} = 1 \\ 1 & \text{if } d_{rr} = 0 \end{cases} \cdots$$

This x is an element in the set T since its decimal digits are 0's or 1's. On the other hand, it differs from d_k at the k^{th} decimal place (which is d_{kk}) for every k which means that x has no place in the enumeration of the elements of T. This impossible situation arose because we assumed that T is countable.

The tiny looking set T of the real numbers between 0 and $1/9 = 0.\overline{1}$ is uncountably infinite and so the full set of all real numbers **R** which is much bigger than T must be uncountable. It follows that the set of all irrational numbers is uncountable since the rationals are countable. It can be seen easily that between any two distinct real numbers, there are uncountably many real numbers.

The set $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ of all pairs of real numbers (x, y) is uncountable because $\mathbf{R}^2$ is the Euclidean plane and it contains the uncountable set **R**, as a subset like the X-axis or the Y-axis or any straight line for that matter.

Remarks: (1) The construction and proof of the uncountability of the set T above can be imitated to show that the set of all sequences formed from the 2 elements 0 and 1 is uncountable, i.e., the set

$$T = \left\{ (x_n)_{n=1}^{\infty} \mid x_n = 0 \text{ or } 1 \text{ for } n = 1,2,\ldots \right.$$

is uncountable.

(2) If $(A_n)_{n=1}^{\infty}$ is a sequence of non-empty finite sets,each having at least 2 elements, then the Cartesian product $\prod_{n=1}^{\infty} A_n$ is uncountable.

(3) The *Cantor set*: A very well-known and popular example of an uncountable subset of the closed interval [0,1] of the real line **R** is the so called *Cantor set* whose construction and properties are of independent interest.[1] It consists ofall real numbers between 0 and 1 whose *trecimal expansion* i.e ternary expansion (i.e., expansion with base 3 instead of the decimal expansion with base 10) contains only the digits 0 and 2.

It is an *uncountable, closed, perfect, nowhere dense set of Lebesgue measure zero,* whatever these words mean!

Density of Rational Points in the Plane

The subset $\mathbf{Q}^2 = \mathbf{Q} \times \mathbf{Q}$ of rational points is *dense* in the plane $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ in the sense that at each point $P=(x,y) \in \mathbf{R}^2$, and any circle C with center P and radius $\varepsilon > 0$, (however small ε may be), there are infinitely many rational points in the inside of C.

As before, this can be expressed as: *every neighbourhood of P in the plane contains infinitely many rational points, however small the neighbourhood may be.*

Pigeonhole Principle

Suppose there are m pigeons to be spread into n chambers. If $m>n$, then whichever way they are distributed, *at least one chamber* will contain 2 or more pigeons. This is called the pigeonhole principle for finite sets. Some natural extensions of this principle are the following:

- Suppose there are infinitely many objects to be spread into finitely many boxes, then at least one box contains infinitely many objects. (Finite union of finite sets is finite)

• Suppose there are uncountably many objects to be spread into cquntably many boxes, then at least one box will contain uncountably many objects. (Countable union of countable sets is countable.)

• Suppose there are countably many objects to be spread into uncountably many boxes, then only countably many boxes can be non-empty. (Uncountable union of disjoint non-empty sets is uncountable.)

Justification of the Model

Assume that the rain is falling on the rational points $\mathbf{Q}^2$ in the plane $\mathbf{R}^2$. Take two dry points P and Q (i.e., $P, Q \in \mathbf{R}^2$ but P, $Q \notin \mathbf{Q}^2$) (*Figure 1*). Consider the set A of all circles in $\mathbf{R}^2$ passing through P and Q. This set A is uncountable, why? There is one and only one circle passing through P, Q and any point not on the line PQ. Or, if L is the perpendicular bisector of the segment PQ, there is one and only one circle passing through P and Q with each point R on L as the center and radius $RP(=RQ)$. The choice for R is uncountable and hence A is uncountable.

Since all these circles are different and pass through two non-rational common points P and Q, if a rational point S lies on one

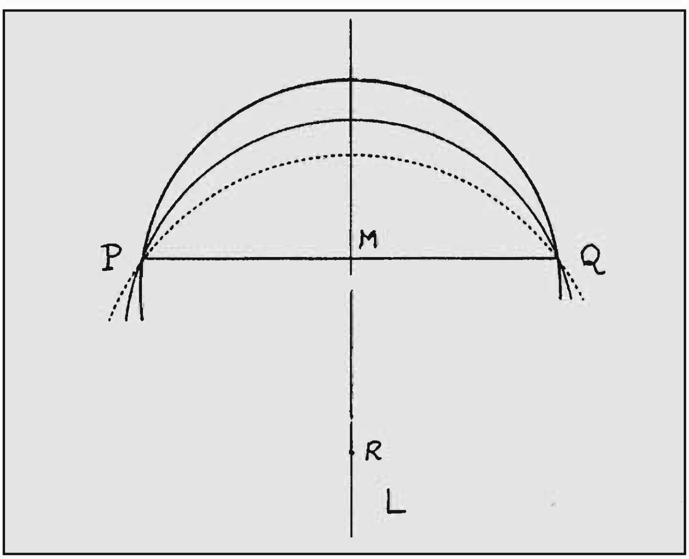

Figure 1

of them, it cannot lie on any other. In other words, a rational point lies on at the most one of these circles. Or equivalently, the set of circles passing through the rational points is countable.

Now apply the pigeonhole principle (3) above for the countable set of rational points $\mathbf{Q}^2$ and the uncountable set A of circles. Ignoring those countably many circles which pass through rational points (i.e., wet points), what is left in A is still uncountable. Nakula can choose any one of these circular arcs to go from P to Q.

(To be able to walk without stepping on or dashing against somebody in the streets of the mega cities like Calcutta or Mumbai, one has to be a Nakula indeed!)

Arcwise Connectedness

A subset X of $\mathbf{R}^n$ is said to be *arcwise connected* if given any two points P and Q in X, there is a *continuous path* (also called an *arc*) lying entirely in X and passing through P and Q. Exactly the same argument as above (replacing $\mathbf{Q}^2$ by Y) gives the following simple result in Topology.

Theorem: The complement $X = \mathbf{R}^n - Y$ of a countable subset Y in $\mathbf{R}^n$ is *arcwise connected* for $n \geq 2$. (The condition that $n \geq 2$ is essential since removing a point from $\mathbf{R}$ breaks the line into two disjoint half rays making it disconnected.)

Suggested Reading

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