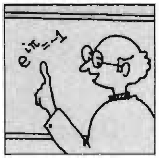
Classroom



In this section of* Resonance, *we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

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An Elementary Problem That Interested Ramanujan!

Ramanujan used to contribute problems/ solutions to the Elementary Problems and Solutions section of the Journal of the Indian Mathematical Society. The following problem appeared in JIMS (Q 289, III 90) and Ramanujan contributed a solution JIMS(IV, 226): to determine the value of the "infinite continued square root"

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+\cdots}}}}}.$$

We proceed as follows.[1] For $x \geq 0$, define $f(x)$ by

$$f(x)=\sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)\sqrt{1+\cdots}}}}.$$

To start with, suppose that $x \geq 1$. Observe that

$$f(x)=\sqrt{1+xf(x+1)}$$
$$\leq \sqrt{(1+x)f(x+1)} \leq \cdots \leq \prod_{i=1}^{\infty}(x+i)^{1/2^i}.$$

[1] The solution given here is different from that given by Srinivasa Ramanujan and follows that given by D J Newman for the same problem, in his book *A Problem Seminar.*

Using the inequalities $x + i \leq 2xi$ for $x, i \geq 1$ and $\prod i^{1/2^i} < \prod 2^{(i-1)/2^i} = 2$, we obtain

$$\prod_{i=1}^{\infty}(x+i)^{1/2^i} \leq$$

$$\prod_{i=1}^{\infty}(2xi)^{1/2^i} = 2x \cdot \left(\prod_{i=1}^{\infty} i^{1/2^i}\right) < 2x \cdot 2 = 4x\,.$$

Thus $f(x) < 4x$ for $x \geq 1$. If $0 < x < 1$ then $x + 1 > 1$, leading to $f(x+1) < 4(x+1)$, and

$$f(x) < \sqrt{1 + x(4x+4)} = 1 + 2x < 1 + 4x\,.$$

It follows that $f(x) \leq 1 + 4x$ for all $x \geq 0$.

The iterative dance now commences. Suppose that for some $a > 0$ we know that $f(x) \leq 1 + ax$ for all $x \geq 0$. Then $f(x+1) \leq 1 + a + ax$, and so

$$\begin{aligned} f(x) &\leq \sqrt{1 + x(1 + a + ax)} \\ &= \sqrt{1 + (a+1)x + ax^2} \\ &\leq 1 + \frac{a+1}{2}x, \end{aligned}$$

since $a \leq \left(\frac{a+1}{2}\right)^2$.

Thus $f(x) \leq 1 + ax$ implies $f(x) \leq 1 + \frac{a+1}{2}x$. Repeating this step, and noting that for any $a > 0$ the sequence a, $\frac{a+1}{2}$, $\frac{a+3}{4}$, $\frac{a+7}{8}, \ldots$ converges to 1, we conclude that $f(x) \leq 1 + x$ for all $x \geq 0$.

Now for the lower bound. Clearly $f(x+1) \geq f(x)$ for all $x \geq 0$, so it follows that $f(x) \geq \sqrt{1 + xf(x)}$. The inequality is easily solved to give $f(x) \geq (x + \sqrt{x^2+4})/2$. We therefore have,

$$f(x) \geq \frac{x}{2} + \sqrt{1 + \frac{x^2}{4}} \geq 1 + \frac{x}{2} \text{ for all } x \geq 0\,.$$

The anchor has been secured and the iteration can begin. Assume that for some $a > 0$, we have $f(x) \geq 1 + ax$ for all $x \geq 0$. We now have,

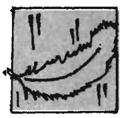
$$\begin{aligned} f(x) &= \sqrt{1 + xf(x+1)} \\ &\geq \sqrt{1 + x(1 + a + ax)} = \sqrt{1 + (1+a)x + ax^2} \\ &\geq 1 + \sqrt{a}x, \text{ since } 1 + a \geq 2\sqrt{a}\,. \end{aligned}$$

Thus $f(x) \geq 1 + ax$ implies $f(x) \geq 1 + \sqrt{a}x$. Repeating this step, and noting that for any $a > 0$, the sequence a, $a^{1/2}$, $a^{1/4}$, $a^{1/8}, \ldots$ converges to 1, we conclude that $f(x) \geq 1 + x$ for all $x \geq 0$.

The sandwich has closed with a pleasing finality about it, and we arrive at the beautiful result: $f(x) = 1 + x$. In particular, the answer to the original problem is $f(2) = 3$.

Corollary: There is precisely one function $f : [0, \infty) \rightarrow [0, \infty)$ which increases monotonically with x and for which $f(x) = \sqrt{1 + xf(x+1)}$ for all $x \geq 0$; namely, the function $f(x) = 1 + x$.

Double Your Money



Tarski belonged to a group of Polish mathematicians who frequented the celebrated Scottish Cafe in Lvov. Another member was Stefan Banach. All sorts of curious ideas came out of bull sessions in the Scottish Cafe. Among them is a theorem so ridiculous that it is almost unbelievable, known as the Banach-Tanski Paradox. It dates from 1924, and states that it is possible to dissect a solid sphere into six pieces, which can be reassembled, by rigid motions, to form two solid spheres each the same size as the original.

But what about the volume? It doubles. Surely that's impossible? The trick is that the pieces are so complicated that they don't have volumes. The total volume can change. Because the pieces are so complicated, with arbitrarily fine detail, you can't actually carry out this dissection on a lump of physical matter. A good job too, it would ruin the gold market.

The Problems of Mathematics, Ian Stewart, Oxford University Press, 1992, pp. 173.