Dimensional Analysis

Keeping Track of Length, Mass, Time ....

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Dimensional analysis is a useful tool which finds important applications in physics and engineering. It is most effective when there exist a maximal number of dimensionless quantities constructed out of the relevant physical variables. Though a complete theory of dimensional analysis was developed way back in 1914 in a seminal paper by Buckingham, systematic procedures necessary to construct a sufficient number of dimensionless quantities have become available only recently. In this article, we describe with an example the steps involved in the Szirtes algorithm which is fairly simple to understand and quite straightforward to use.

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Dimensional analysis is a topic which every student of science encounters in elementary physics courses. The basics of this topic are taught and learnt quite hurriedly (and forgotten fairly quickly thereafter!) It does not generally receive the attention and the respect it deserves even though it has wide applications in all branches of physics and engineering. In the field of physics proper, it is most commonly used as a tool for checking equations for dimensional correctness. (The requirement that all the terms in an equation describing a physical law should have the same dimensions seems to have been first stated by Fourier.) Furthermore, given certain inputs based on experimental observations or data, dimensional analysis can be used to derive empirical laws which would otherwise be quite difficult, if not impossible, to arrive at.

In the field of engineering, dimensional analysis plays a more important role in addition to the above mentioned applications.

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Quite often, it is very essential to construct and study a scaled-down model of the system to be investigated or a machine to be constructed. For example, prediction of the erosion rate of the river banks would be practically impossible without constructing model structures which properly incorporate the essential qualitative features of a given river system. In aerodynamics, wind turbulence effects on aircraft dynamics is most conveniently studied by using scaled-down models of the full-size systems. In machine design and construction, model studies can significantly reduce the possibility of simple but costly errors in the construction of the final version, generally called the 'prototype'. It is, of course, important that one determines the 'scaling factors' which relate the parameters or the variables of the model to those of the prototype. It is here that the real power of dimensional analysis comes into the picture quite explicitly.

Any important application of dimensional analysis requires the construction of 'dimensionless quantities'. Given a set of physical variables necessary to describe the problem at hand, one constructs by suitable combinations, (by way of multiplications or divisions) quantities where all the dimensions cancel out completely. Dimensionless quantities are constructed and used in almost all the branches of physics and engineering. While the number of such quantities in any given field are generally quite small, there are a few branches of physics where it is almost impossible to discuss any problem without using an appropriate dimensionless quantity. In this connection, it may be interesting to note that fluid mechanics probably tops the list with a total of about 44 dimensionless quantities! As an example of a typical dimensionless quantity, consider the flow of a fluid in a pipe or a channel, which is best described in terms of the fluid density ρ, the pipe radius or the channel width a, the flow velocity u and the fluid viscosity η. Then, the quantity $R \equiv \rho a u/\eta$ (called the Reynolds number) is dimensionless and plays a very important role in determining the transition of the flow from the laminar to the turbulent state. The advantage of using dimensionless quantities is that although their number is generally much smaller than that of the actual physical variables, if chosen properly they would be adequate to characterize certain aspects of system dynamics, such as the flow transition in the example above.

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The π-Theorem

A general theory of dimensional analysis and its implications was developed way back in 1914 in a seminal paper by Buckingham. The main result of his work is summarized in a well-known theorem, which is generally referred to as the π-*theorem*. The theorem is applicable to any dimensionally homogeneous equation which relates, say, n physical quantities defined in terms of r reference dimensions (such as M, L and T). A physical equation is said to be dimensionally homogeneous if every term in the equation reduces to the same algebraic quantity when expressed in terms of the reference dimensions. According to the central result of the π-theorem, it is always possible to reduce the equation to a relationship between $(n-r)$ independent dimensionless quantities provided the reference dimensions themselves are considered as independent of one another. The minimal set of such dimensionless quantities for a given system constitutes the fundamental or the complete set. A formal algebraic proof of the Buckingham's π-theorem has been discussed by Isaacson and Isaacson, while Corrsin has given an elegant proof based on geometrical considerations. A number of results of practical importance follow as a consequence of the π-theorem, which have been further discussed in the work done by Bridgman, Ipsen, Duncan and Pankhurst (suggested reading).

While the π-theorem requires the existence of a complete set of dimensionless quantities, there is no unique or universally applicable method to actually construct such quantities explicitly. In his original paper, Buckingham has indicated the basic outline of a procedure that follows as a consequence of the π-theorem. There are, however, practical difficulties in its actual implementation for finding out a minimal set of dimensionless quantities which would not only be appropriate for the problem at hand but also sufficient to describe unambiguously the behavior of the system under consideration. Isaacson and Isaacson have discussed a couple of methods based on the construction of a relevant set of indicial equations. Recently, Thomas Szirtes of SPAR Aerospace, Canada has given a procedure which is better suited for direct application to a number of physical problems, even when the number of physical variables is large. The Szirtes algorithm is fairly simple and quite straightforward, and is formulated in terms of results from the theory of matrices. In the following, I summarize this procedure and illustrate the various steps involved by taking the example of fluid flow in a channel.

The Szirtes Algorithm

According to Szirtes, the following steps are involved in the construction of a minimal set of dimensionless quantities:

Step 1

Given the system, identify the variables, parameters and constants which govern its behavior. It is most essential that all the relevant quantities are included in the list since if any one of them is omitted, the outcome of the dimensional analysis could be erroneous. However, if any irrelevant or superfluous quantity is included, it does not influence the outcome but only makes the analysis more complicated. Thus, when in doubt about the suitability of a quantity, it is best to include it in the list!

As an example, I will consider in the following the flow of a fluid in a pipe or a channel and construct the familiar Reynolds number. Clearly, as mentioned above, the flow would depend on the parameters ρ, a, u, and η. While one can verify *a posteriori* that just these variables are enough to construct the Reynolds number, let us include, say, the surface tension α which may be important in certain flow problems. The number of dimensionless quantities to be constructed would depend on the number of reference

Table 1

	α	ρ	a	u	η
M	1	1	0	0	1
L	0	–3	1	1	–1
T	–2	0	0	–1	–1

dimensions employed. Taking the familiar CGS system with three reference dimensional units (M denoting the mass, L the length and T the time), tabulate the various variables and the units as given in *Table 1.*

The entries in *Table 1* are just the exponents of the corresponding dimensions for each of the variables. For example, viscosity has the dimensions $ML^{-1}T^{-1}$, hence the corresponding values 1, –1, and –1 in the η-column in *Table 1.*

Step 2

Form the square matrix A by taking the right most elements of *Table 1*. Obviously, the order of this matrix would be equal to the number (m) of reference dimensions used. In *Table 1*, we have three reference dimensions, namely, M, L and T and therefore, the matrix A is of the order 3×3. Make sure that the matrix A is non-singular, that is, $\det A \neq 0$. If $\det A = 0$, rearrange the columns in *Table 1* so as to obtain a new square matrix with non-zero determinant. Call the matrix formed by the remaining elements in *Table 1* as matrix B which need not be square.

Thus, for the example of fluid flow, we have

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & -1 \end{bmatrix}, \quad \mathrm{B} = \begin{bmatrix} 1 & 1 \\ 0 & -3 \\ -2 & 0 \end{bmatrix}$$

Step 3

Calculate the inverse of the matrix A and find the product matrix C defined by

$$C = -(A^{-1}B)^T,$$

where T denotes the transpose operation (that is, the interchange of the rows with the corresponding columns) and A^{-1} is the inverse of the matrix A.

For the example under consideration, we can easily show that

$$A^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$$

Step 4

Extend *Table 1* as described in the following. Place the matrix C below the matrix A. Place an identity (square) matrix I of appropriate size below the matrix B. Extend the column containing M, L, T by the dimensionless quantities (to be constructed) denoted by, say, π_i, $i = 1, 2, 3, \ldots$ Thus, if the number of physical variables listed is n and the number of reference dimensions used is m with $n > m$, then the dimension of the matrix A is $m \times m$, that of B is $m \times (n-m)$, that of C is $(n-m) \times m$ and that of I is $(n-m) \times (n-m)$. According to the Buckingham π-theorem, the number of dimensionless quantities that can be constructed is given by $N_d = n - m$. Denoting the $m \times 1$ column matrix consisting of the reference dimensions (like M, L and T) as the matrix D and the $(n-m) \times 1$ column matrix consisting of the dimensionless quantities (namely, $\pi_1, \pi_2, \ldots, \pi_{n-m}$) as the matrix π, we have schematically the *Table 2*, where the $1 \times n$ row matrix V has as elements the various physical variables (like, α, ρ, a, u, and η) used to describe the system. The dimensions of each of the matrices is shown just below the corresponding entry in *Table 2*. Clearly, the total matrix formed by A, B, C, and I is of

Table 2

MATRIX [DIMENSIONS]	V $[1 \times n]$	
D $[m \times 1]$	B $[m \times (n-m)]$	A $[m \times m]$
π $[(n-m) \times 1]$	I $[(n-m) \times (n-m)]$	C $[(n-m) \times m]$

dimension $n \times n$. Each of the dimensionless quantities π_i, $i=1, 2, 3, \ldots, (n-m)$ is then constructed from *Table 2* by forming the product of all the listed physical variables after taking the corresponding entries in the table as their exponents.

For our example, we obtain the output in *Table 3.*

The dimensionless quantities π_1 and π_2 are then easily read off from the *Table 3* as

$$\pi_1 = \alpha^1 \rho^0 a^0 u^{-1} \eta^{-1} = \alpha/u\eta,$$

$$\pi_2 = \alpha^0 \rho^1 a^1 u^1 \eta^{-1} = \rho a u/\eta.$$

Table 3

	α	ρ	a	u	η
M	1	1	0	0	1
L	0	−3	1	1	−1
T	−2	0	0	−1	−1
π_1	1	0	0	−1	−1
π_2	0	1	1	1	−1

Note that the dimensionless quantity π_2 is just the Reynolds number mentioned in the beginning!

It is clear from the above discussion that if the entry α had been omitted in *Table 1*, then the analysis would have yielded only one dimensionless quantity, namely, the Reynolds number (π_2). On the other hand, inclusion of α leads to the existence of an additional variable π_1 which may be useful in certain flow problems where fluid surface tension plays an important and explicit role. Likewise, one could add extra variables to the list which would yield the corresponding dimensionless quantities. For example, if the acceleration due to gravity (g) is included, we then obtain a third dimensionless quantity given by $\pi_3 = ag/u^2$.

Scaling Laws

The theorem established by Buckingham ensures that the set of dimensionless quantities constructed according to the above algorithm unambiguously describes the behavior of the physical system under consideration. They can then be used in the construction of models as well as for deriving the scaling laws. A model is said to be dimensionally similar to the prototype if the value of every dimensionless quantity is the same for both. For the above example, if the subscripts m and p denote, respectively, the corresponding quantities for the model and the prototype, for dimensional similarity, we must have

$$\frac{\alpha_m}{u_m \eta_m} = \frac{\alpha_p}{u_p \eta_p}, \quad \frac{\rho_m a_m u_m}{\eta_m} = \frac{\rho_p a_p u_p}{\eta_p},$$

which after rearrangements yield

$$\frac{\alpha_m}{\alpha_p} = \frac{u_m}{u_p} \cdot \frac{\eta_m}{\eta_p}, \quad \frac{\rho_m}{\rho_p} \cdot \frac{a_m}{a_p} = \frac{\eta_m}{\eta_p} \cdot \left(\frac{u_m}{u_p}\right)^{-1}$$

Denoting the various ratios by S (with subscripts corresponding to the various physical variables), we finally obtain the scaling laws,

$$S_\alpha = S_u\, S_\eta\,,\quad S_\rho S_a = S_\eta S_u^{-1}.$$

These two relations should be satisfied in order that the model and the prototype are dimensionally similar. It should, however, be noted that even though the two systems are dimensionally similar, they could be geometrically quite dissimilar.

Discussion

Let me now describe briefly two uses of dimensionless quantities and the scaling laws mentioned earlier from which one may derive important conclusions about a system and its model. (For illustration, I take the example of fluid flow discussed above). First, with certain inputs, the qualitative behavior of a given system can be understood easily. For the above example, the input is the experimental fact that beyond a certain value of the Reynolds number (called the critical Reynolds number R_c which is generally in the range of about 2000) the fluid flow undergoes a transition from the laminar to the turbulent state. Defining the kinematic viscosity $\nu=\eta/\rho$, we have $R=au/\nu$. Therefore, for a given pipe radius a, fluids with larger kinematic viscosity need larger flow speeds in order to undergo the transition. On the other hand, for the same (incompressible) fluid, the onset of turbulent flow occurs at smaller flow speeds for larger pipe radius. Such conclusions which may appear to be obvious can be made quantitative by using the dimensionless quantities. Second, as mentioned above, scaling laws are most important in the construction of models for prototypes. For the above example, if we take the same fluid in the prototype and in the model, then $S_p = 1$, $S_\eta = 1$ and we obtain the simple scaling law $S_a S_u = 1$. Hence, if the scale-size of the model is halved, that is, $S_a = 1/2$, then, for dimensional similarity, the flow speed should be doubled.

Finally, let me conclude by pointing out another important application of dimensionless quantities in a slightly different context. They can be used for simplifying the graphical presentation of experimentally obtained data or theoretically derived equations. Take the example of the Reynolds number. Generally, it is possible to show in one chart in two dimensions, in the simplest way, the functional dependence of three quantities. For example, for a given value of the pipe radius a, we can plot in the $\eta - u$ plane lines of constant density ρ. From the structure of these lines, one can figure out the onset of turbulent flow. However, if such information is needed for a number of different values of a, one requires that many charts. On the other hand, in terms of the Reynolds number R the onset is characterized by just one number, namely, the critical Reynolds number R_c. In fact, if we plot the relation $R = \rho a u/\eta$ in the $R - u$ plane, we obtain a straight line whose slope is just the ratio $\rho a/\eta$. Given the value of R_c, it is then very easy to display graphically the velocity ranges for which the flow is laminar/turbulent and also the transition velocity.

Thus, the topic of 'dimensional analysis' does not seem to be all that trivial!!

An Application to Simulation of Ionospheric 'Heating'

The upper regions of earth's atmosphere at distances of 80 kms and beyond contain matter in a (weakly) ionized state called the *plasma state*. The *ionosphere* not only acts as a protective shell for life on earth but has also played a significant role in making long distance telecommunication a reality. The different layers of the ionosphere act like mirrors for the electromagnetic waves sent from the ground based radio-wave transmitters and reflect them back to the receivers on the ground, thus making 'wireless' communication possible. Needless to say that ionospheric plasma is one of the most intensely studied naturally occurring media surrounding our planet.

Much of our knowledge about the earth's ionosphere has been derived by means of 'passive' experiments carried out over the decades using ground-based, balloon, rocket and, more recently, satellite techniques. On the other hand, with the development of high-power transmitters, it has become possible since the early 1970's to conduct *active* experiments, particularly in the F-region altitudes (150 – 500 kms); that is, as in laboratory experiments, it is possible to induce controlled changes in a small volume of the ionospheric plasma and then study its response to external stimuli, (see Fejer, Leyser and Rao, Kaup in Suggested Reading). Typically, such experiments are carried out using strong electromagnetic (pump) waves in the frequency range ~2 –15 Mhz sent from ground-based transmitters. The pump electromagnetic waves heat the ionospheric plasma in local regions and thereby lead to significant changes in the plasma temperature and number density.

The next natural step is to be able to simulate the ionospheric conditions in laboratory experiments which can be easily controlled. This requires a knowledge of the proper scaling laws between the various variables which characterize the ionospheric system and the simulated model. Hence, it is necessary to identify the relevant dimensionless quantities, and this can be easily done using the Szirtes algorithm as follows:

The ionospheric plasma is characterized by the number density n, the temperature θ, the electron-ion collision frequency ν_{ei} and the ambient terrestrial magnetic field B. The incident pump wave is characterized by the wave frequency ω and the input power P. For a proper sizing of the simulated model, we also need a scale-length variable L. Thus, the matrix V containing the various physical variables which characterize the system is given by,

$$V = [B \quad n \quad \omega \quad P \quad \theta \quad \nu_{ei} \quad L\,] \,.$$

For convenience, let us introduce the temperature θ as an additional reference dimension in addition to M, L and T. We then have the matrices,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ -3 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -3 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

- A simple calculation shows that,

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 0 & -1 & 0 \\ -2 & 1 & 0 & 0 \end{bmatrix}, \quad C = -(A^{-1}B)^T = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 3 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Table 2 can now be easily constructed for the present example as,

	B	n	ω	P	θ	ν_{ei}	L
M	$\frac{1}{2}$	0	0	1	0	0	0
L	$-\frac{1}{2}$	–3	0	2	0	0	0
T	–1	0	–1	–3	0	–1	0
θ	0	0	0	0	1	0	0
π_1	1	0	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{3}{2}$
π_2	0	1	0	0	0	0	3
π_3	0	0	1	0	0	–1	0

Finally, the three dimensionless quantities π_1, π_2 and π_3 are then easily read off from the above table as,

$$\pi_1 \equiv B^1 P^{-1/2} \nu_{ei}^{1/2} L^{3/2} = \frac{B\nu_{ei}^{1/2} L^{3/2}}{P^{1/2}},$$

$$\pi_2 \equiv n^1 L^3 = nL^3,$$

$$\pi_3 \equiv \omega\nu_{ei}^{-1} = \frac{\omega}{\nu_{ei}},$$

It is easy to recognize the physical content of the quantities π_2 and π_3. The quantity π_2 is simply the total number of plasma particles in a cube of linear dimension L, and is similar to the *plasma parameter*

$(n\lambda^3_D)^1$ while π_3 is just the ratio of the wave frequency to the collision frequency. The meaning of π_1 can be made clear by considering

$$\pi_1^2 = \frac{\left(B_0^2 L^3\right)\nu_{ei}}{P},$$

which is essentially the ratio of the total magnetic field energy per unit collision period to the pump wave power. As earlier, one can now easily derive the scaling laws and thus determine the necessary parameter values for carrying out a scaled down simulation of the ionospheric heating process.

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[1] The symbol λ_D is called the *Debye length*. In plasma physics, if one considers a volume much larger than $\lambda_D{}^3$, the Coulomb forces enforce near-equality of the number of positive and negative charges.

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