Fourier Series

The Mathematics of Periodic Phenomena

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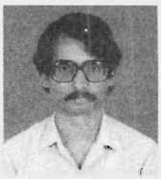

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There are several natural phenomena that are described by periodic functions. The position of a planet in its orbit around the sun is a periodic function of time; in chemistry, the arrangement of molecules in crystals exhibits a periodic structure. The theory of Fourier series deals with periodic functions. By a periodic function we mean a function f of a real variable t which satisfies the relation $f(t+c) = f(t)$ for all t, where c is constant. Here c is called a *period* of f. The simplest examples of periodic functions of period 2π are provided by the trigonometric functions $\sin kt$ and $\cos kt$ for integer k. It follows that linear combinations of the form

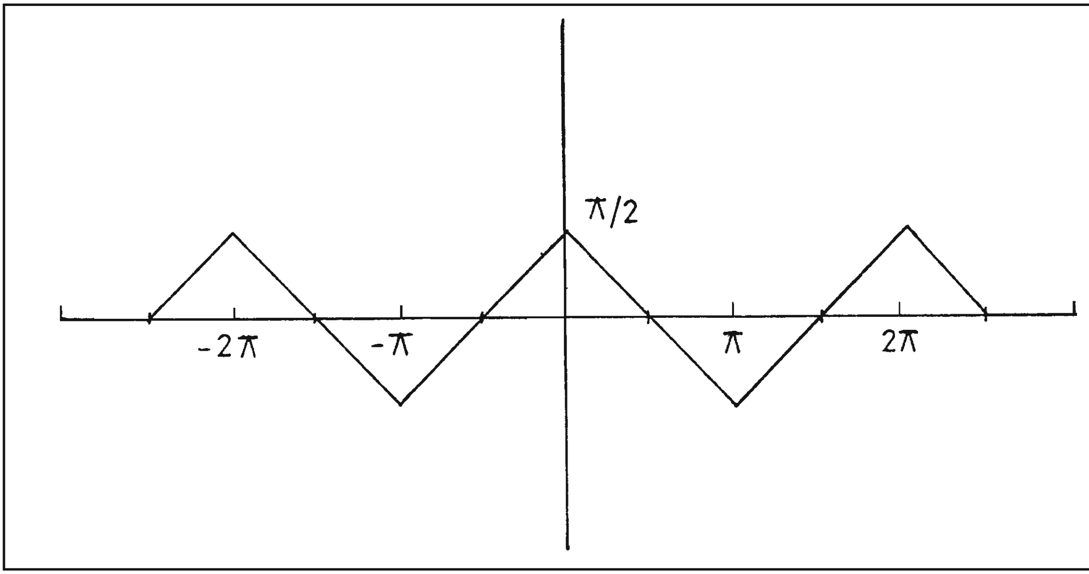
$$P_n(t) = \sum_{k=0}^{n} (a_k \cos kt + b_k \sin kt)$$

are also periodic with the same period 2π. These are called *trigonometric polynomials*. The basic idea behind Fourier series is that any periodic function (of period 2π) can be approximated by trigonometric polynomials. (Notice that it is enough to consider periodic functions with period 2π because a function f with some other period, say c, can be converted to a function of period 2π by a suitable change of variable. (Exercise !)) The subject of Fourier series finds a wide range of applications from crystallography to spectroscopy. It is one of the most powerful theories in the history of mathematics and has stimulated the development of several branches of analysis.

Consider the function $f(t)$ defined by the formula

$$f(t) = \begin{cases} \frac{\pi}{2} - t, & 0 \le t \le \pi \\ \frac{\pi}{2} + t, & -\pi \le t \le 0 \end{cases}$$

Figure 1



which for other values of t is extended by periodicity. Thus the graph of the function looks as shown in *Figure 1*.[1]

This function is continuous but not differentiable at the points $0, \pm\pi, \pm 2\pi, \cdots$ where the graph has corners. The function $f(t)$ is even. The functions $\cos kt$ are all even and the functions $\sin kt$ are all odd.[2] Hence we may expect to approximate f by trigonometric polynomials of the form $\sum_{k=0}^{l} a_k \cos kt$. In fact consider

$$P_n(t) = \frac{4}{\pi}\sum_{k=0}^{n}(2k+n)^{-2}\cos(2k+1)t.$$

The graphs of these functions for $n=2$ and for $n=4$ are shown in *Figure 2*.

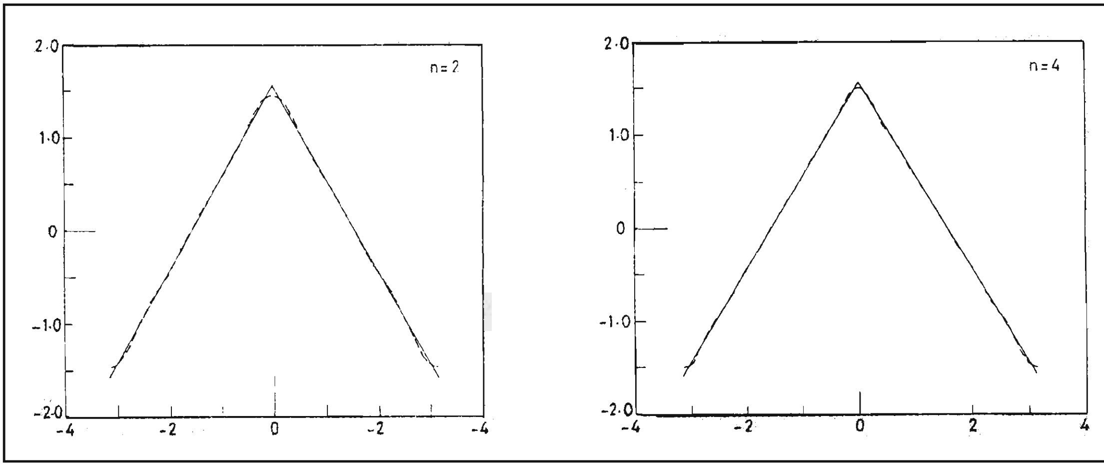
From the picture it is clear that even for small values of n, the trigonometric polynomial P_n approximates f very well.

Now you may be wondering why we have chosen the particular trigonometric polynomials P_n in order to approxi-mate f. As we have already remarked the absence of the sine terms may be

[1] The author thanks A S Vasudevamurthy for helping him with the various graphs in this article. The material in some of the boxes and footnotes were provided by C Varughese.

[2] A function h is *even* if $h(x) = h(-x)$ for all x and *odd* if $h(x) = -h(-x)$ for all x.

Figure 2



explained by the fact that f is even and $\sin kt$ are all odd; but why did we choose the coefficients $(2k+n)^{-2}$ for the terms $\cos(2k+1)t$? Is there a way of relating the function f to the coefficients $(2k+n)^{-2}$? To answer all these questions, let us go back in history by a couple of centuries. The basic idea that any periodic function may be expanded in terms of trigonometric polynomials is attributed to D Bernoulli. This Swiss mathematician believed in the 'universal character' of the trigonometric polynomials much before Fourier. It all started with his works in 1747, 1748 and 1753 where he studied the vibrating string problem: find the solutions of the following partial differential equation with the given initial conditions

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2}$$

$$u(x,0) = f(x), \quad u(-\pi,t) = u(\pi,t) = 0$$

This is called the *wave equation* and the graph of $u(x, t)$ represents the shape of the vibrating string at time t. The initial shape of the string is given by the function f and the condition $u(-\pi, t) = 0 = u(\pi, t)$ means that the string is kept fixed at the ends.

This equation had been studied by Euler and D'Alembert before Bernoulli. D' Alembert gave the solution of the above problem in the form

$$u(x,t) = \frac{1}{2}(f(x+t) + f(x-t)).$$

We can easily check that this function u is indeed a solution provided f is an odd periodic function with period 2p and is twice differentiable. But Bernoulli had a different idea. The functions

$$u_k(x,t) = a_k \cos kt \sin kx$$

satisfy the wave equation and following physical ideas Bernoulli suggested solutions of the form

$$u(x,t) = \sum_{k=0}^{\infty} a_k \cos kt \sin kx.$$

Based on this observation Bernoulli was led to believe in the possibility of expanding arbitrary periodic f with $f(\pi) = f(-\pi) = 0$ in terms of $\sin kx$ but he did not have a clue as to how to calculate the various coefficients!

Mathematics had to wait for almost fifty years for a formula to calculate the *Fourier coefficients*. In 1807, the French mathematician Jean Joseph Fourier working on the heat equation found a way to calculate the coefficients.

The idea of Fourier is every simple. Let us rewrite the trigonometric polynomials

$$P_n(t) = \sum_{k=0}^{n} (a_k \cos kt + b_k \sin kt)$$

in the form

$$P_n(t) = \sum_{k=-n}^{n} \alpha_k e^{ikt}.$$

That these two are equivalent follows from the well known relation

$$e^{ikt} = \cos kt + i \sin kt, \text{ where } i = \sqrt{-1}\,.$$

Suppose that these trigonometric polynomials $P_n(t)$ converge to $f(t)$ in any reasonable sense so that we can take the limit under the integral sign to get

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} P_n(t)e^{-imt}dt = \int_{-\pi}^{\pi} f(t)e^{-imt}dt, \text{ for each fixed } m.$$

The integrals on the left hand side of the above equation can be easily calculated. In fact

$$\int_{-\pi}^{\pi} P_n(t)e^{-imt}dt = \sum_{k=-n}^{n} \alpha_k \int_{-\pi}^{\pi} e^{i(k-m)t}dt$$

and an easy integration reveals that

$$\int_{-\pi}^{\pi} e^{i(k-m)t}dt = 0$$

whenever $k \neq m$. And when $k=m$ we get 2π for the value of the integral. Thus we have the formula for α_m:

$$\alpha_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-imt}dt.$$

To emphasize that α_m depends on f we write $\alpha_m = \hat{f}(m)$. The associated series

$$f(t) = \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikt}$$

is called the *Fourier series* of the function f.[3] $\{\hat{f}(k)\}_{k=-\infty}^{\infty}$ are called the *Fourier coefficients*.

[3] The equality here must be taken to mean that the partial sums of the series on the right approximate the function f in a sense that will be described later. The two sides are not necessarily equal for every value of t, although it is customary to write it in this form.

(Exercise: If f is differentiable and f' is continuous show that the Fourier series corresponding to f' is given by $\sum_{k=-\infty}^{\infty}(ik)\hat{f}(k)\exp(ikt)$.)

Fourier's idea is so simple that one may wonder why it eluded D'Alembert, Bernoulli, and even the great Euler. But one has to bear in mind that these mathematicians of the seventeenth century had to deal with concepts like functions, integrals and the convergence of infinite series when they were not well understood. For Euler, a function always meant an analytic expression; Bernoulli believed that any possible position of a vibrating string represents a function and can be given by an analytic expression. It was Fourier who went one step further to consider all functions either continuous or discontinuous and boldly asserted that they can be represented by an infinite series which came to be known as Fourier series. The moral of the story is: a simple argument can still be very deep and it may not be very simple to discover!

Let us go back and calculate the Fourier coefficients of the function f which we started with. As f is even

$$\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt$$

$$= \frac{1}{\pi}\int_{0}^{\pi}\left(\frac{\pi}{2} - t\right)dt = 0$$

For $k \neq 0$,

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cos ktdt$$

since the integral

$$\int_{-\pi}^{\pi} f(t)\sin ktdt = 0$$

on account of the fact that $\sin kt$ is odd. Therefore

$$\hat{f}(k) = \frac{1}{\pi}\int_{0}^{\pi}\left(\frac{\pi}{2} - t\right)\cos ktdt$$

$$= -\frac{1}{\pi}\int_{0}^{\pi} t\cos ktdt$$

since the integral $\int_0^{\pi} \cos kt\, dt = 0$. Integrating by parts in the last integral we can easily see that

$$\hat{f}(k) = \frac{1}{\pi k^2}(1 - \cos k\pi) = \frac{1}{\pi k^2}(1 - (-1)^k),$$

the last equality holds because $\cos k\pi = (-1)^k$. Thus $\hat{f}(2k) = 0$ and $\hat{f}(2k+1) = \frac{2}{\pi}(2k+1)^{-2}$ and consequently the Fourier series of f becomes

$$f(t) = \frac{2}{\pi}\sum_{k=-\infty}^{\infty}(2k+1)^{-2} e^{i(2k+1)t}$$

Combining the terms corresponding to $(2k+1)$ and $-(2k+1)$ we get

$$f(t) = \frac{4}{\pi}\sum_{k=0}^{\infty}(2k+1)^{-2}\cos(2k+1)t.$$

Now, it should be clear why the polynomials $P_n(t)$ gave good approximations to the function $f(t)$!

Thanks to Fourier, we now have a way of writing down the Fourier series of any periodic function. But what is the guarantee that the Fourier series actually represents the function? In other words how do we know if the partial sums defined by

$$S_n f(t) = \sum_{k=-n}^{n} \hat{f}(k) e^{ikt}$$

converge to the function f pointwise, i.e. how do we know that $S_n f(t)$ converges to $f(t)$ as $n \to \infty$, for every t? It is not very hard to prove that for the particular function of *Figure 1*, we have such convergence. *Figure 2* is convincing evidence of this fact. Fourier believed and asserted that, in general, $S_n f$ always converges to the function f, but he offered no proof. *Fourier's assertion turned out to be wrong as we will see shortly.* This by no means diminishes the greatness of Fourier's contributions. In fact, with a different interpretation of convergence his assertion is correct: the Fourier

Box 1

If we do not insist on pointwise convergence, but are willing to settle for *mean square convergence*, then indeed for a wide class of 2π–periodic functions f, the Fourier series converges to f. All we need to assume is that f is *square-integrable* i.e. $\int_{-\pi}^{\pi} |f(x)|^2\, dx$ is finite. For such f, $S_n f$ converges to f in the mean-square sense even if for *many* values of t, $S_n f(t)$ does *not* converge to $f(t)$ i.e. $\frac{1}{2\pi}\int_{-\pi}^{\pi} |S_n f(t) - f(t)|^2 dt \to 0$ as $n \to \infty$ i.e. the *average value* of the square of the error $|S_n f(t) - f(t)| \to 0$ as $n \to \infty$. If f represents a periodic signal, physicists and electrical engineers will tell you that $\int_{-\pi}^{\pi} |f(t)|^2\, dt$ represents the *energy* in the signal. Using the fact about mean square convergence explained above, we can actually prove that $\int_{-\pi}^{\pi} |f(t)|^2\, dt = 2\pi \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2$. However instead of using $\exp(ikt)$, if we use $\sin kt$ and $\cos kt$ and write the Fourier series as $a_0 + \sum_{k=1}^{\infty} b_k \cos kt + \sum_{k=1}^{\infty} c_k \sin kt$, then the above formula can be written as $\int_{-\pi}^{\pi} |f(t)|^2\, dt = 2\pi |a_0|^2 + \pi \sum_{k=1}^{\infty} |b_k|^2 + \pi \sum_{k=1}^{\infty} |c_k|^2$ (why?). Finally if g is another 2π – periodic function with corresponding Fourier series $A_0 + \sum_{k=1}^{\infty} B_k \cos kt + \sum_{k=1}^{\infty} C_k \sin kt$, then one has the more general formula $\int_{-\pi}^{\pi} f(t) g(t) dt = 2\pi a_0 A_0 + \pi \sum_{k=1}^{\infty} b_k B_k + \pi \sum_{k=1}^{\infty} c_k C_k$.

series of a function converges in what the modern day mathematicians call *the sense of distributions*.

The theory of distributions, an important development of this century, was actually anticipated by Fourier as one can see from his treatment of Fourier integrals! For another mode of convergence, which is valid for a very wide class of functions, see *Box 1*.

The pointwise convergence of the partial sums $S_n f$ to the function f fails — it fails dramatically at some points as we see from the following example. Consider the function $g(t) = -1$ for $-\pi \le t \le 0$, $g(t) = 1$ for $0 < t \le \pi$. This function is discontinuous at the points 0, $\pm\pi$, $\pm 2\pi$, ..., where it has a jump of size 2. We can easily calculate the Fourier series of this function. We leave it as an exercise to the reader to show that $\hat{g}(0) = 0, \hat{g}(2k) = 0$ and that $\hat{g}(2k+1) = -i\frac{2}{\pi}(2k+1)^{-1}$. Therefore, the Fourier series of g takes the form

$$g(t) = \frac{-2i}{\pi} \sum_{k=-\infty}^{\infty} (2k+1)^{-1} e^{(2k+1)it}$$

or combining the terms with $(2k+1)$ and $-(2k+1)$ we can write it as

$$g(t) = \frac{4}{\pi} \sum_{k=0}^{\infty} (2k+1)^{-1} \sin(2k+1)t.$$

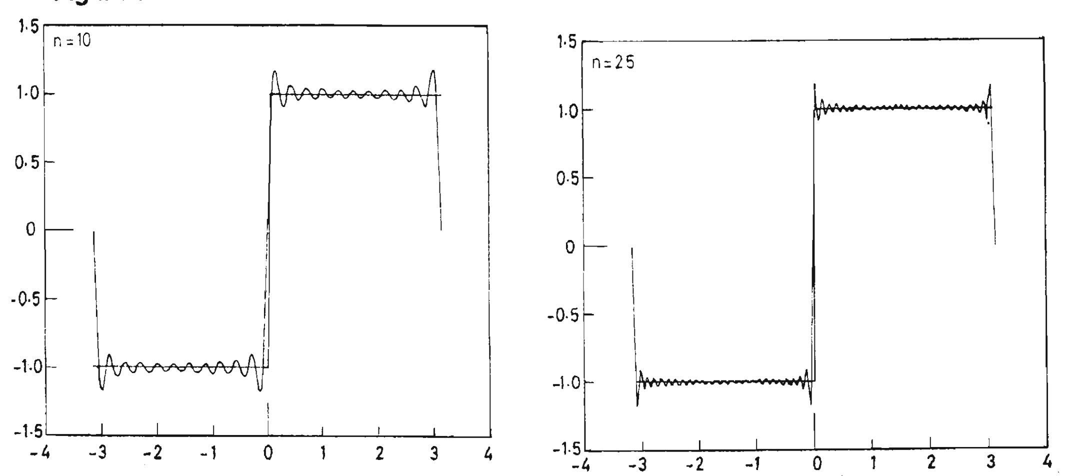
Consider now the partial sums associated to the above Fourier series:

$$S_n g(t) = \frac{4}{\pi} \sum_{k=0}^{n} (2k+1)^{-1} \sin(2k+1)t.$$

The graphs of some of these trigonometric polynomials are shown in *Figure 3*.

From the graphs we observe that $S_n g$ approximates g well except in a small neighbourhood of $t=0$ where it overshoots and under-

Figure 3



shoots the levels ± 1 respectively. Naturally one expects that the overshoot and undershoot tends to zero as n goes to infinity; but surprisingly this does not take place. Instead, they are always about 9%. This is called the *Gibbs phenomenon* because it was pointed out by Gibbs in a letter to *Nature* in 1899. The letter was a reply to one from Michelson (of the *Michelson - Morley experiment* fame), who apparently was angry with the machine he used for computing Fourier series, as it failed badly at jumps!

Box 2

Let $f(t)=t$, $-\pi \leq t<\pi$, and extended for other values of t by periodicity. Use the result at the end of *Box 1* to calculate the value of π^2 as the sum of an infinite series.

The Gibbs phenomenon that occurs in the vicinity of jump discontinuities can be proved rigorously. This already shows that we cannot expect the Fourier series to converge at all points. One may think that the situation will be different in the case of continuous functions. But already in 1876, du Bois - Reymond constructed a continuous function whose Fourier series fails to converge at a given point. Several eminent mathematicians including Dirichlet, Riemann and Cantor occupied themselves with the problem of convergence of Fourier series. Many positive results were proved in the course of time but in 1926 the Russian

Box 3

Notice that the functions $\sin kt$ and $\cos kt$ are periodic with period $(2\pi/k)$. Of course they are periodic with period 2π but they are also periodic with the smaller period $(2\pi/k)$ and this is the smallest period for these functions. Thus, they have *frequency* $(k/2\pi)$. By choosing various values of k, we get what we can think of as a *standard family* of signals. If f is a periodic signal (with period 2π), its Fourier series is a decomposition of f into signals from the standard family. The relation $\int_{-\pi}^{\pi}|f(t)|^2 dt = 2\pi\sum|\hat{f}(k)|^2$ (see *Box 1*) is then a description of how the energy contained in the signal f is distributed among various frequencies. The relative amount of energy contained at any frequency is determined by the square of the modulus of the corresponding Fourier coefficient.

mathematician A N Kolmogorov came up with a discouraging result. He constructed an integrable (in the sense of Lebesgue) function whose Fourier series diverges everywhere ! After this example by Kolmogorov, mathematicians started believing that one day someone will construct a continuous function with everywhere divergent Fourier series. That day never came! Instead, in 1966, the Swedish mathematician L Carleson showed that in the case of continuous functions the Fourier series can fail to converge only on a set which is *negligible* in a certain sense (i.e. of *measure* zero)!

When the function f is smooth, say once differentiable and f' is continuous, it can be shown that the Fourier series converges to the function at each point, and even *uniformly*. But in applications we have to deal with functions having very bad discontinuities. Dirichlet and Dini found local conditions on the functions that will ensure pointwise convergence of the Fourier series. These results are adequate for the purpose of applications. However mathematicians not being content with such results have been seeking more and more general results proving the convergence of Fourier series for larger and larger classes of functions! They have also worked with different notions of convergence although unfortunately many of these cannot be explained here without assuming a lot of Lebesgue theory of integration.

We would like to conclude this article with the following result of Fejer which treats the class of continuous functions as a whole. As we know, given any point t_0 there is a function in this class whose Fourier series diverges at that point. In 1904, the Hungarian mathematician Fejer had the brilliant idea of considering the *averages of partial sums instead of the partial sums themselves.* Thus he considered

$$\sigma_n f = \frac{1}{n+1}\left(S_0 f + S_1 f + \cdots S_n f\right).$$

These are called the *Cesaro means* and it is easy to see that they can be written in the form

$$\sigma_n f(t) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) \hat{f}(k) e^{ikt}.$$

The celebrated theorem of Fejer says that when f is continuous the Cesaro means $\sigma_n f$ converge to f, not only pointwise, but also in the stronger sense of *uniform convergence*. Hence trigonometric polynomials are *dense* in the class of continuous functions !

The body of literature dealing with Fourier series has reached epic proportions over the last two centuries. We have only given the readers an outline of the topic in this article. For the full length episode we refer the reader to the monumental treatise of A Zygmund. Beginners will find the books of Bhatia and Körner very helpful. For more advanced readers, the books of Dym - McKean, Folland, Helson and Katznelson make enjoyable reading. The article of Gonzalez - Velasco deals with the influence of Fourier series on the development of analysis and we highly recommend it.

Suggested Reading

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