Planets Move in Circles !

A Different View of Orbits

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The orbits of planets, or any other bodies moving under an inverse square law force, can be understood with fresh insight using the idea of velocity space. Surprisingly, a particle moving on an ellipse or even a hyperbola still moves on a circle in this space. Other aspects of orbits such as conservation laws are discussed.

Yes, it is true. And no, it is not the cheap trick of tilting the paper to see an ellipse as a circle. The trick, as you will see, is a bit more sophisticated. It turns out that the trajectory of a particle, moving under the attractive inverse square law force, is a circle (or part of a circle) in the *velocity space* (The high-tech name for the path in velocity space is *hodograph*). The proof is quite straightforward. Start with the text book result that, for particles moving under any central force $f(r)\,\hat{\mathbf{r}}$, the angular momentum $\mathbf{J}=\mathbf{r}\times\mathbf{p}$ is conserved. Here $\mathbf{r}$ is the position vector, $\mathbf{p}$ is the linear momentum and $\hat{\mathbf{r}}$ is the unit vector in the direction of $\mathbf{r}$. This implies, among other things, that the motion is confined to the plane perpendicular to $\mathbf{J}$. Let us introduce in this plane the polar coordinates (r, θ) and the cartesian coordinates (x, y). The conservation law for $\mathbf{J}$ implies

$$\frac{d\theta}{dt} = \text{constant}/r^2 \equiv hr^{-2}, \qquad (1)$$

which is equivalent to Kepler's second law, since $(r^2\dot{\theta}/2) = h/2$ is the area swept by the radius vector in unit time. Newton's laws of motion give

$$m\frac{dv_x}{dt} = f(r)\cos\theta;\; m\frac{dv_y}{dt} = f(r)\sin\theta\,. \qquad (2)$$

Dividing (2) by (1) we get

$$m\,\frac{dv_x}{d\theta} = \frac{f(r)\,r^2}{h}\cos\theta;\; m\,\frac{dv_y}{d\theta} = \frac{f(r)\,r^2}{h}\sin\theta\,. \qquad (3)$$

The high-tech name for the path in velocity space is *hodograph*.

The miracle is now in sight for the inverse square law force, for which $f(r)\,r^2$ is a constant. For planetary motion we can set it to $f(r)\,r^2 = -GMm$ and write the resulting equations as

$$\frac{dv_x}{d\theta} = \frac{-GM}{h}\cos\theta;\; \frac{dv_y}{d\theta} = \frac{-GM}{h}\sin\theta\,. \qquad (4)$$

Integrating these equations, with the initial conditions $v_x\,(\theta=0) = 0$; $v_y\,(\theta=0) = u$, squaring and adding, we get the equation to the *hodograph*:

$$v_x^2 + (v_y - u + GM/h)^2 = (GM/h)^2 \qquad (5)$$

which is a circle with center at $(0, u - GM/h)$ and radius GM/h. So you see, planets do move in circles!

Some thought shows that the structure depends vitally on the ratio between u and GM/h, motivating one to introduce a quantity e by defining $(u - GM/h) \equiv e\,(GM/h)$. The geometrical meaning of e is clear from *Figure 1*. If $e=0$, i.e, if we had chosen the initial conditions such that $u = GM/h$, then the center of the hodograph is at the origin of the velocity space and the magnitude of the velocity remains constant. Writing $h=ur$, we get $u^2 = GM/r^2$ leading to a circular orbit in the real space as well. When $0<e<1$, the origin of the velocity space is inside the circle of the hodograph. As the particle moves the magnitude of the velocity changes between a maximum of $(1+e)\,(GM/h)$ and a minimum of $(1-e)\,(GM/h)$. When $e=1$, the origin of velocity space is at the circumference of the hodograph and the magnitude of the velocity vanishes at this point. In this case, the particle goes from a finite distance of closest approach, to

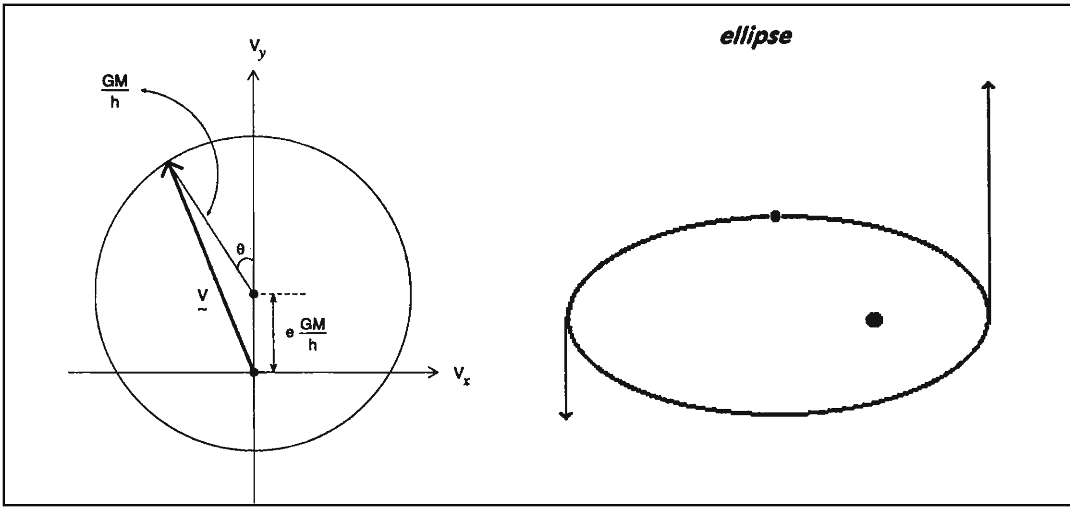

Figure 1 Velocity Space: The x and y components of the velocity vector v are plotted. The origin represents zero velocity, and the circle gives the velocity of the planet at different times i.e it is the orbit in velocity space. The radius vector in real space is parallel to the tangent to this circle, because changes in velocity are parallel to the force which is central. The angle θ turned by the tangent is thus the same as that turned by the radius vector. Real space orbit at top right.

infinity, reaching infinity with zero speed. Clearly, $e = 1$ implies $u^2 = 2GM/r_{initial}$ which is just the text book condition for escape velocity and a parabolic orbit.

When $e > 1$, the origin of velocity space is outside the hodograph and *Figure 2* shows the behaviour in this case. The maximum velocity achieved by the particle is OB when the particle is at the point of closest approach in real space. The asymptotic velocities of the particle are OA and OC obtained by drawing the tangents from O to the circle. From the figure it is clear that $\sin\phi = e^{-1}$. During the *unbound motion* of the particle, the velocity vector traverses the part ABC. It is circles all the way! (Incidentally, can you find a physical situation in which the minor arc AC could be meaningful?)

Given the velocities, it is quite easy to get the real space trajectories. Knowing $v_x(\theta)$ and $v_y(\theta)$ from (4) one can find the kinetic energy as a function of θ and equate it to $(E+GMm/r)$, thereby recovering the conic sections. Except that, there is a more elegant way of doing it.

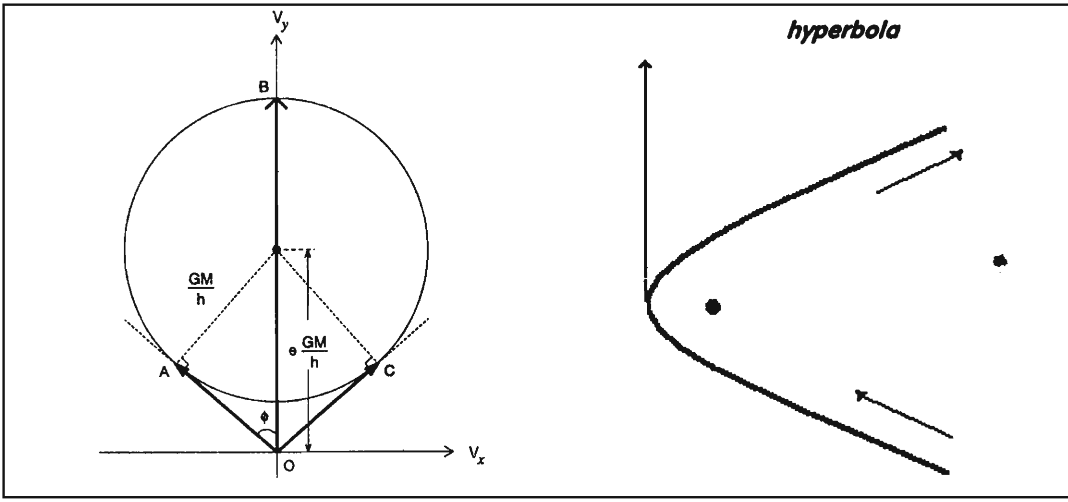

Figure 2 Velocity space representation of a hyperbolic orbit. Note that the origin is now outside the circle. Only the arc ABC is traversed. OB is the maximum velocity, attained at closest approach, 2ϕ is the angle of scattering. Real space orbit shown at right.

There is a theorem proved by Newton through (rather than *in*) Principia, which states that 'anything that can be done by calculus can be done by geometry' and our problem is no exception. The geometrical derivation is quite simple: In a small time interval δt, the magnitude of the velocity changes by $\Delta v = \hat{\mathbf{r}}\,(GM/r^2)\,\delta t$ according to Newton's law. The angle changes by $\Delta\theta = (h/r^2)\,\delta t$ from the conservation of angular momentum. Dividing the two relations, we get

$$\frac{\Delta |v|}{\Delta\theta} = \frac{GM}{h} \qquad (6)$$

But in velocity space Δv is the arc length and $\Delta\theta$ is the angle of turn and if the ratio between the two is a constant, then the curve is a circle. So there you are.

To get the real space trajectory from the hodograph, we could reason as follows: Consider the transverse velocity v_T at any instant. This is clearly the component perpendicular to the instantaneous radius vector. But in the central force problem, the velocity change Δv is parallel to the radius vector. So v_T is also perpendicular to Δv; or in other words, the v_T must be the component of velocity parallel to the radius vector in the velocity space. *Voila!* From *Figure 1*, it is just

$$v_T = \frac{GM}{h} + \frac{GM}{h} e\cos\theta = \frac{GM}{h}(1+e\cos\theta) = \frac{h}{r} \qquad (7)$$

with the last relation following from the definition of angular momentum. One immediately sees the old friend — the conic section — with a latus rectum of $l=h^2/GM$ and eccentricity of e (Good we didn't denote the ratio between $(u-GM/h)$ and (GM/h) by k or something!).

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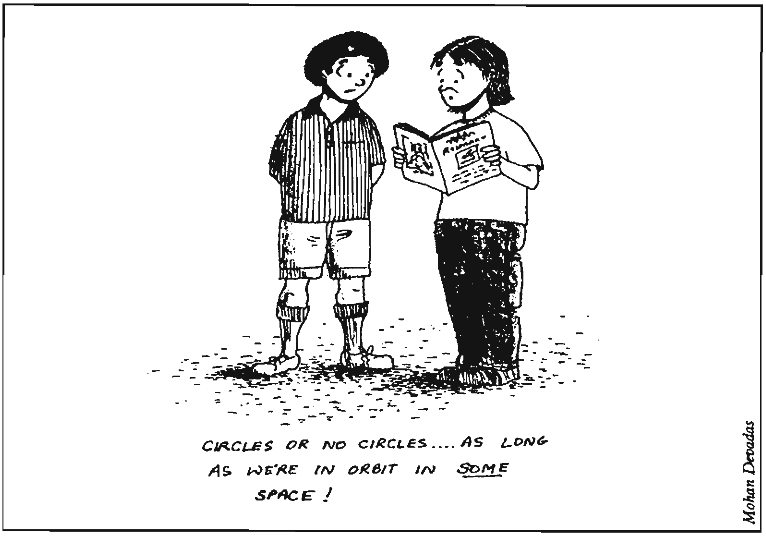
The elegance of geometry over calculus in the above analysis (or anywhere for that matter, though lots of people disagree) is a bit fake with calculus entering through the back door. But even with calculus, the more general way to think about the Kepler problem is as follows. For any time-independent central force, we have constancy of energy E and angular momentum **J**. Originally, a particle moving in 3 space dimensions has a phase space[1] which is 6 dimensional. Conservation of the four quantities (E, J_x, J_y, J_z) confines the motion to a region of $6-4=2$ dimensions. The projection of the trajectory onto the xy-plane will, in general, fill

[1] This is a space obtained by combining the three coordinates xyz with three momentum components mv_x, mv_y, $m\,v_z$. This is a good way of describing the current state of the system, since one can use this information to predict the future.



a two-dimensional region of space. That is, the orbit should fill a finite region of the space in this plane, if there are no other conserved quantities. But we are always taught that the bound motion is an ellipse in the xy-plane, which is an one-dimensional curve. So, there must exist yet another conserved quantity for the inverse square law force which keeps the planet in one dimension rather than two. And indeed there is, which provides a really nice way of solving the Kepler problem.

To discover this last constant, consider the time derivative of the quantity $\mathbf{p} \times \mathbf{J}$ in any central force $f(r)\,\hat{\mathbf{r}}$. With a little bit of algebra, you can show that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{p} \times \mathbf{J}) = -\, m f(r)\, r^2 \frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t} \tag{8}$$

The miracle of inverse square force is again in sight: When $f(r) r^2$ = constant $= -GMm$, we find that the vector

$$\mathbf{A} \equiv \mathbf{p} \times \mathbf{J} - \frac{GMm^2}{r}\mathbf{r} \tag{9}$$

is conserved. But we needed only one more constant of motion, now we have got three which will prevent the particle from moving at all! No, it is not an overkill; one can easily show that A satisfies the following relations:

$$A^2 = 2mJ^2E + (GMm)^2; \ \mathbf{A} \cdot \mathbf{J} = 0\,. \tag{10}$$

The first one tells you that the magnitude of $\mathbf{A}$ is fixed in terms of other constants of motion and so is not independent; and the second shows that $\mathbf{A}$ lies in the orbital plane. These two constraints reduce the number of independent constants in $\mathbf{A}$ from 3 to 1, exactly what we needed. It is this extra constant that keeps the planet on a sensible orbit (i.e. a closed curve!). To find that orbit, we only have to take the dot product of (9) with the radius vector $\mathbf{r}$ and use the identity $\mathbf{r} \cdot (\mathbf{p} \times \mathbf{J}) = \mathbf{J} \cdot (\mathbf{r} \times \mathbf{p}) = J^2$. This gives

Stepping into Kepler's shoes

You must have read that Kepler analysed the astronomical data of Tycho Brahe and arrived at his laws of planetary motion. Ever wondered how exactly he went about it? Remember that the observations are made from the Earth which itself moves with an unknown trajectory! Suppose you were given the angular positions of all the major astronomical objects over a long period of time, obtained from some fixed location on Earth. This is roughly what Kepler had. How will you go about devising an algorithm that will let you find the trajectories of the planets ? Think about it!

$$\mathbf{A}\cdot\mathbf{r} = Ar\cos\theta = J^2 - GMm^2 r \qquad (11)$$

or, in more familiar form,

$$\frac{1}{r} = \frac{GMm^2}{J^2}\left(1 + \frac{A}{GMm^2}\cos\theta\right). \qquad (12)$$

As a bonus we see that **A** is in the direction of the major axis of the ellipse. One can also verify that the offset of the centre in the hodograph, $(GM/h)\,e$, is equal to (A/h). Thus **A** also has a geometrical interpretation in the velocity space. It all goes to show how special the inverse square law force is! If we add a component $1/r^3$ to the force, (which can arise if the central body is not a sphere) J and E are still conserved but not **A**. If the inverse cube perturbation is small, it will make the direction of **A** slowly change in space and we get a 'precessing' ellipse.

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Suggested Reading

- Sommerfeld A. Lectures on Theoretical Physics. *Mechanics*. Academic Press. Vol.I, p.40
- Rana N C and Joag P S. *Classical Mechanics*. Tata McGraw Hill. p.140

Poisson Mathematics

That Poisson liked teaching can be seen from his own words: "Life is made beautiful by two things — studing mathematics and teaching it". (From: *The Mathematical Intelligencer*, Vol.17, No.1, 1995)